4^0. A new research framework in Approximation Theory, Numerical Analysis, and Computational Mathematics.

This entire formulation is called the **Computational (Numerical) Diameter (C(N)D)** and is divided into three sequential tasks:

C(N)D-1. Reconstruction from exact information, depending on the type of functionals and algorithms for processing the numerical information obtained from them, encompasses all known approximation theory, numerical analysis, computational mathematics, and function theory – Fourier series, bases.

C(N)D-2. In the optimal computational aggregate, the values of the informational functionals can be replaced by nearby values while maintaining optimality. The search for the largest such deviations forms an independent problem of finding extremal errors – a new optimization problem.

C(N)D-3. There exists, or does not exist, another computational aggregate with a structure analogous to that of the considered optimal computational aggregate, and perhaps even more general, but with a larger order of extremal error.

The *C(N)D* problem is historically and substantively presented in full detail in the articles

Temirgaliyev N., Zhubanysheva A.Zh. Teoriya priblizheniy, Vychislitel'naya matematika i Chislennyy analiz v novoy kontseptsii v svete Komp'yuternogo (vychislitel'nogo) poperechnika [Approximation Theory, Computational Mathematics and Numerical Analysis in new conception of Computational (Numerical) Diameter], Vestnik Evraziyskogo natsional'nogo universiteta imeni L.N. Gumileva. Seriya Matematika. Informatika. Mekhanika [Bulletin of L.N. Gumilyov Eurasian National University. Mathematics, computer science, mechanics series]. 2018. Vol. 124. \No 3. P. 8-88.

Temirgaliyev N., Zhubanisheva A.Zh. Computational (Numerical) Diameter in a Context of General Theory of a Recovery, Russian Mathematics. 2019. Vol. 63. \No 1. P. 79-85.

The problem of recovery from exact information (C(N)D-1) has been known and developed over many decades in numerous publications with various notations. These works include those by A. Sard, S.M. Nikolsky, N.S. Bakhvalov, I. Babushka and S.L. Sobolev, C.A. Mitchelli and T. J. Ravlin, J.F. Traub, G. Vasilkovsky, and H. Wozhniakovsky, A. Pincus, V.N. Temlyakov, N.P.Korneichuk, E. Novak, H. Vozhnyakovsky, J. Vibiral, B. Kashin, E. Kosov, I. Limonova, V. Temlyakov, and many others.

Regarding recovery from inexact information, the publications can be divided into three groups: the first group includes works by N. Temirgaliev and co-authors; the second group – by V.M. Tikhomirov, G.G. Magaril-Ilyaev, K.Y. Osipenko, A.G. Marchuk and co-authors; and the third group – by J. Traub, H. Vozhnyakovsky, L. Plaskota and co-authors. These groups differ not only in their problem statements but also, interestingly, in the terminology used for essentially the same mathematical objects, and consequently in the formulation of results. The problem of recovery from inexact information in the context of the second and third parts (C(N)D-2 and C(N)D-3) represents a new development.

The next lower bound, obtained for all possible computational aggregates constructed from arbitrary linear information $l_1(f),\dots,l_N(f)$ and processed according to the algorithm $\varphi_N(z_1,\dots,z_N;\cdot)$, without any restrictions except for natural ones

$$\inf_{\substack{l_1,\dots,l_N-\text{всевозможные}\\ \text{линейныефункционалы};\varphi_N}} \sup_{f\in W_p^r(0,1)^s} \left\| f^{(\alpha_1,\dots,\alpha_s)}(\cdot)-\varphi_N(l_1(f),\dots,l_N(f);\cdot)\right\|_{L^q(0,1)^s}$$
$$\gg \begin{cases} N^{-\frac{r}{s}+\frac{\alpha_1+\dots+\alpha_s}{s}+\left(\frac{1}{p}-\frac{1}{q}\right)}, \text{если} 2\le p\le q\le\infty,\\ N^{-\frac{r}{s}+\frac{\alpha_1+\dots+\alpha_s}{s}+\frac{1}{2}-\frac{1}{q}}, \text{если} 1\le p< 2\le q\le+\infty,\\ N^{-\frac{r}{s}+\frac{\alpha_1+\dots+\alpha_s}{s}}, \text{если} 1\le p\le q<2 \end{cases}$$

allows the widely developed C(N)D-1 problem, with its separately studied cases, to be unified into a single framework through upper bounds that encompass the known diameters of Kolmogorov, Korneichuk, Tikhomirov, Temlyakov, etc., and approximation problems – Fourier-Lebesgue series over all possible orthonormal systems and their means, various bases and frames. This also covers actively researched topics of recent years, including wavelet systems and greedy algorithms.

The C(N)D formulation in three tasks will be illustrated using the example of Numerical Differentiation in one (out of three possible) case $2\le p\le q\le\infty$, $r>\alpha_1+\dots+\alpha_s+s\left(\frac{1}{p}-\frac{1}{q}\right)$, $N=2^{ns}$ $(n=1,2,\dots)$, where the function $(\alpha_1,\dots,\alpha_s)$ −derivatives are defined in the sense of Weyl."

C(N)D-1: $\delta_N(0;L_N(W_p^r(0,1)^s)\times\{\varphi_N\}_{L^q(0,1)^s})_{L^q(0,1)^s}\equiv$

$$\equiv \inf_{\substack{l_1,\dots,l_N-\text{всевозможные}\\ \text{линейныефункционалы};\phi_N}} \sup_{f\in W_p^r(0,1)^s} \left\| f^{(\alpha_1,\dots,\alpha_s)}(\cdot)-\varphi_N(l_1(f),\dots,l_N(f);\cdot)\right\|_{L^q(0,1)^s}$$
$$\succ\prec N^{-\frac{r}{s}+\frac{\alpha_1+\dots+\alpha_s}{s}+\left(\frac{1}{p}-\frac{1}{q}\right)},$$

Here, the optimal (non-improvable) order of recovery from exact information has been obtained.

C(N)D -2: For $(\alpha_1,\dots,\alpha_s)$ derivative of the partial sum of Fourier-Lebesque series

$$\bar{\varphi}\left(\{\hat{f}(m)\}_{m\in I_n};x\right)\equiv S_N^{(\alpha_1,\dots,\alpha_s)}(f;x)=\sum_{\substack{m=(m_1,\dots,m_s)\in Z^s:\\ |m_j|\le 2^n(j=1,\dots,s)}}\hat{f}(m)\prod_{j=1}^{s}\left(e^{2\pi i m_j x_j}\right)^{(\alpha_j)}$$

the value $\tilde{\varepsilon}_N\equiv\tilde{\varepsilon}_N(\Phi_N)\equiv\tilde{\varepsilon}_N\left(\Phi_N(W_p^r)\times\{\varphi_N\}_{L^q(0,1)^s}\right)=N^{-\frac{r}{s}-\left(1-\frac{1}{p}\right)}$ is the limiting error:

Firstly,

$$\delta_N(0;L_N(W_p^r(0,1)^s)\times\{\varphi_N\}_{L^q(0,1)^s})_{L^q(0,1)^s}\succ\prec$$

$$\succ\prec \delta_N(\tilde{\varepsilon}_N(\Phi_N)=N^{-\frac{r}{s}-\left(1-\frac{1}{p}\right)};L_N(W_p^r(0,1)^s)\times\{\varphi_N\}_{L^q(0,1)^s})_{L^q(0,1)^s}\equiv$$

$$\equiv \inf_{\substack{l_1,\dots,l_N-all\ possible\\ linear\ functionals;\ \varphi_N}} \sup\left\{\left\| f^{(\alpha_1,\dots,\alpha_s)}(\cdot) - \varphi_N\left(l_1(f)+\gamma_N^{(1)}\tilde{\varepsilon}_N,\dots,l_N(f)+\gamma_N^{(N)}\tilde{\varepsilon}_N;\cdot\right)\right\|_{L^q(0,1)^s} : f\in W_p^r(0,1)^s, \left|\gamma_N^{(\tau)}\right| \le 1(\tau=1,\dots.,N)\right\} \succ\prec$$

$$\succ\prec \delta_N(\tilde{\varepsilon}_N(\Phi_N) = N^{-\frac{r}{s}-\left(1-\frac{1}{p}\right)}; S_N^{(\alpha_1,\dots,\alpha_s)}(f;x))_{L^q(0,1)^s} \equiv$$

$$\equiv \sup_{\substack{f\in W_p^r(0,1)^s\\ \left|\gamma_N^{(\tau)}\right|\le 1(\tau=1,\dots,N)}} \left\| f^{(\alpha_1,\dots,\alpha_s)}(x) - \sum_{\substack{m=(m_1,\dots,m_s)\in Z^s:\\ |m_j|\le 2^n (j=1,\dots,s)}} \left(\hat{f}(m)+\gamma_N^{(m)}\tilde{\varepsilon}_N\right)\prod_{j=1}^{s}\left(e^{2\pi i m_j x_j}\right)^{(\alpha_j)}\right\|_{L^p(0,1)^s} \succ\prec$$

$$\succ\prec N^{-\frac{r}{s}+\frac{\alpha_1+\dots+\alpha_s}{s}+\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

secondly, for any increasing to $+\infty$ positive sequence $\{\eta_N\}_{N=1}^{+\infty}$ there is the equality

$$\lim_{N\to\infty}\frac{\delta_N\left(\eta_N\tilde{\varepsilon}_N; \sum_{\substack{m=(m_1,\dots,m_s)\in Z^s\\ |m_j|\le 2^n(j=1,\dots,s)}}\hat{f}(m)\prod_{j=1}^{s}\left(e^{2\pi i m_j x_j}\right)^{(\alpha_j)}\right)_{L^q(0,1)^s}}{\delta_N\left(0; L_N(W_p^r(0,1)^s)\times\{\varphi_N\}_{L^q(0,1)^s}\right)_{L^q(0,1)^s}} = +\infty.$$

Here, the exact order of the maximal error in computing the information functionals has been determined, while preserving the non-improvable orders of recovery from exact information (depending, we emphasize, on the optimal computational aggregate in C(N)D-1, of which there may be more than one).

C(N)D-3: Any computational aggregate $\varphi_N\left(\hat{f}(m^{(1)}),\dots,\hat{f}(m^{(N)});\ x\right)$ constructed by trigonometric Fourier coefficients with an arbitrary spectrum of N harmonics cannot have a limiting error larger (by order) than the limiting error of $S_N^{(\alpha_1,\dots,\alpha_s)}(f;x)$: for any positive sequence increasing to $+\infty$ the following equality is fulfilled

$$\lim_{N\to\infty}\frac{\delta_N\left(\eta_N\tilde{\varepsilon}_N; \varphi_N\left(\hat{f}(m^{(1)}),\dots,\hat{f}(m^{(N)});x\right)\right)_{L^q(0,1)^s}}{\delta_N\left(0; L_N(W_p^r(0,1)^s)\times\{\varphi_N\}_{L^q(0,1)^s}\right)_{L^q(0,1)^s}} = +\infty.$$

Here, the question is addressed whether one can switch to another computational aggregate while achieving order improvements in both recovery speed and maximal error, with the answer being negative.

The approach follows the principle of smoothness of the considered objects, ranging from finitely differentiable and analytic ones to completely non-smooth cases, accompanied by the construction of new mathematical structures endowed with certain analogues of developed topologies.