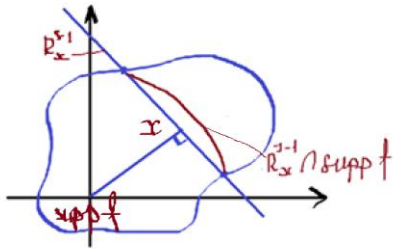
3 Computed tomography in an equivalent two-line linkage with the Computational (numerical) diameter (C(N)D) in the supervised machine learning (ML) framework, leading to artificial intelligence (AI)

The Radon transform of a function $f(x)$is a real-valued function defined on $\mathbb{R}^s$.



$$(Rf)(x) := \int_{y \in R_x^{s-1} \cap supp\, f} f(y)dy =: R\big(f(y)\big)(x),$$

where $R_x^{(s-1)}$is the $(s-1)$-dimensional hyperplane perpendicular to the vector $0x$and passing through the point x(see Figure 1).

Figure 1

The function $f(x)$can be expressed in terms of its Radon transform $(Rf)(x)$. Let $\Omega = [0,1]^s$, and suppose $f \in C_0^\infty(\Omega)$and $f(x) = 0$for all $x \in \mathbb{R}^s \setminus \operatorname{supp} f$. Then, for $x \in \Omega$, $x = (\theta, t) \in S^{s-1} \times R^1$

$$f(x) = \int\limits_{\Omega} f(x)dx + {\sum_k}' \left(\frac{1}{|k|} \sum_{m=-\infty}^{\infty} Rf\left(\frac{k}{|k|}, \frac{m + k \cdot x}{|k|}\right) - \int\limits_{\Omega} f(x)dx \right),$$

where

$$\theta = \frac{k}{|k|}, t = \frac{m + k \cdot x}{|k|},$$

and ${\sum_k}'$ denotes summation over all $k \in \mathbb{Z}^s$such that $k_1 > 0$and the greatest common divisor of $k_1, \dots, k_s$equals 1.

It is clear that a computer operating only with finite information cannot process this functional series as an infinite object. Thus, the problem is to construct a computational device which, with any prescribed accuracy, determines the desired density $f$at any point $x$from a finite set of values $(Rf)(\xi_1), \dots, (Rf)(\xi_N)$.

In the technical implementation of computer tomography (CT), the most common approach is the preliminary selection of scanning points $\xi_1, \dots, \xi_N$, followed by the construction of an approximation theory for the target function $f(x)$based on this data. In this case, the simplest methods are, naturally, parallel and fan-beam scanning (see Figure 2).

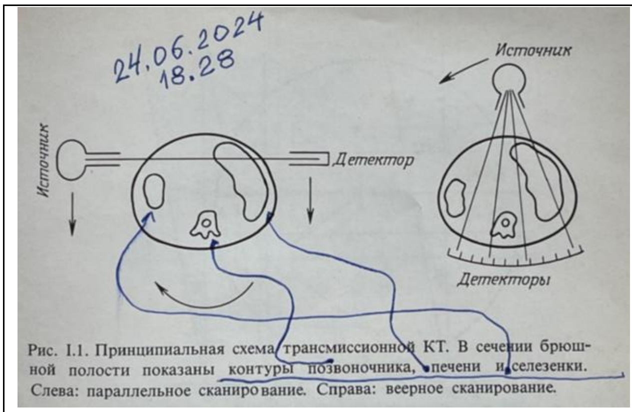


Figure 2

At the same time, from the standpoint of theoretical mathematics, the problem arises of optimally choosing the scanning nodes together with a corresponding algorithm for processing the obtained numerical information, which would allow one, in a given metric and with any prescribed accuracy, to recover the desired density $f(x)$at any point $x$of the considered body. Within the framework of C(N)D, this is precisely the problem C(N)D-1.

In the construction of a technical device for computer tomography, a crucial role is played by C(N)D-2—the quantity of limiting error, which makes it possible to recover the desired function $f(x)$without loss of accuracy. In its absence, the accuracy of the device must be determined experimentally, which involves significant financial costs.

The problem $K(B)P$-3describes which computational systems are not better than the optimal one identified in $K(B)P$-1.

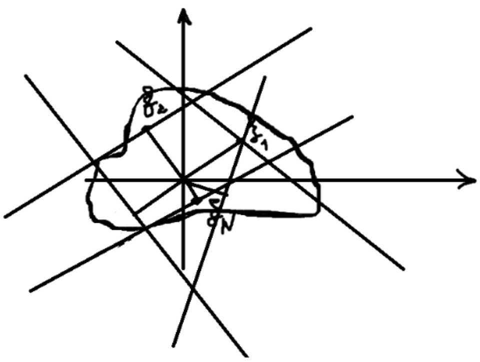


Figure 3

The problem of computer tomography turned out to be unexpectedly equivalent to the problem, developed in approximation theory, of optimal reconstruction of functions via finite convolution (see Figure 3):

$$\sup_{\|f\|_{W_2^{\alpha(y)}(E_s)}\le 1} \left\| f(x) - \sum_{k=1}^{N} f(\xi_k)\,\Phi_N(x-\xi_k) \right\|_{W_2^{\rho(y)}(E_s)} \asymp$$

$$\asymp \sup_{\|f\|_{W_2^{\alpha(y)\cdot\|y\|^{\frac{s-1}{2}}}(S^{s-1}\times R^1)}\le 1} \left\| f(x) - \sum_{k=1}^{N} (R^2)^{-1} R f(\xi_k)\, R\big(\Phi_N(y - \xi_k)\big)(x) \right\|_{W_2^{\rho(y)\cdot\|y\|^{\frac{s-1}{2}}}(S^{s-1}\times R^1)},$$

where the kernel $\Phi_N(x)$is a real-valued function that is 1-periodic in each of the $s$variables $x=(x_1,\dots,x_s)$.

“Contribution of Scientific Theories and Discoveries to the Progress of Society as a Whole (University of Vienna, 2016)”:

The Rector of the University of Vienna (referring to J. Radon) noted: “Mathematical theories are often presented in abstract form, perhaps regarded as sterile tricks, which suddenly turn out to be valuable tools for physical knowledge and thereby unexpectedly reveal their hidden power.”

K. Sigmund stated: “Johann Radon studied abstract problems of so-called pure mathematics and had no idea that today the Radon transform is the basis of computed tomography. Its numerous applications confirm the rule: there is nothing more practical than a good theory.”

In light of these considerations, the present result—expressed in just two lines (!)—claims a strengthening that encompasses essentially all works stemming from Radon's foundational 1917 paper up to the present day, including the theoretical foundations of the devices developed by A. Cormack and G. Hounsfield, who were awarded the Nobel Prize in Medicine in 1979, and indeed all journals across any scientometric scale.

The same statement in an equivalent form:

$$f(x)\overset{B}{=}\sum_{k=1}^{N} R^{-1}f(\xi_k)\, R\big(\Phi(y-\xi_k)\big)(x)+\alpha_N(B)$$

$$\left(x\in S^{s-1}\times R^1;\, B=W_2^{\rho(y)\cdot(y_1^2+\dots+y_s^2)^{\frac{s-1}{2}}}(S^{s-1}\times R^1)\right) \tag{*}$$

in the language of fundamental mathematics, this demonstrates a remarkable transition from theory to practice: on the left we have the function $f(x)$as a "rule f" (implicit and abstract), according to which each $x$is assigned a value $f(x)$, while on the right we obtain an explicit formula that approximates $f(x)$with an error not exceeding $|\ \alpha_N\ |$.

In this sense, the explicit formula (*), written as

$$\sum_{k=1}^{N} R^{-1}f(\xi_k)\, R\big(\Phi(y-\xi_k)\big)(x)\overset{B}{=}f(x)+\alpha_N(B),$$

represents a statement of the theory of "supervised machine learning": it enables the computation of the values of the function $f$at all points $x$based on the data

$$(\xi_1, Rf(\xi_1)), \dots, (\xi_N, Rf(\xi_N)),$$

with maximal error

$$\sup\{|\alpha_N(x)|: x\in S^{s-1}\times R^1\}.$$

Thus, as a computationally theoretical (CT) solution, this framework effectively gives rise to what is now called artificial intelligence.

Practical recommendations of an innovative nature, in the context of automatic response of a computational system to the smoothness of the approximated function, are as follows: according to the mathematical model of the examined object, it is proposed to use direct formulas of computer scanning, in which, based on a prescribed accuracy from the Reconstruction Theorem, the number of nodes at which scanning must be performed is determined. The corresponding formula is thereby transformed into an algorithm that, with the specified accuracy, yields the desired density of the object.

The following table illustrates the transition from the problem of optimal function reconstruction via finite convolution to the reconstruction of the same function via Radon transforms.

1^0. The Sobolev and Nikolsky–Besov classes — scanning is performed at the nodes of a uniform grid:
$\frac{1}{n^s} \sum_{\substack{k=(k_1,\dots,k_s) \\ k_j=1,\dots,n\ (j=1,\dots,s)}} f\left(\frac{k}{n}\right) \sum_{\substack{m=(m_1,\dots,m_s)\in Z^s \\ -n\le m_j\le n\ (j=1,\dots,s)}} exp\left(2\pi i\left(m, y-\frac{k}{n}\right)\right)$
$\frac{1}{n^s} \sum_{\substack{k=(k_1,\dots,k_s) \\ k_j=1,\dots,n\ (j=1,\dots,s)}} (R^2)^{-1}Rf\left(\frac{k}{n}\right) \sum_{\substack{m=(m_1,\dots,m_s)\in Z^s \\ -n\le m_j\le n\ (j=1,\dots,s)}} \int_{y\in R_x^{s-1}} exp\left(2\pi i\left(m, y-\frac{k}{n}\right)\right) dy$
2^0. Classes of Korobov, Sobolev, and Nikolsky–Besov–Amanov type—Kairat Sherniyazov grids in the form of a linear combination of Korobov grids.
$\sum_{\substack{\tau\in Z^s: \\ \tau_j>0, \lVert\tau\rVert<n}} \frac{1}{detA_{n,\tau}} \sum_{k\in K(n,\tau)} f\left(k\left(A_{n,\tau}^{-1}\right)'\right) \sum_{m\in\rho(\tau)} exp\left(2\pi i\left(m, y-k\left(A_{n,\tau}^{-1}\right)'\right)\right)$
$\sum_{\substack{\tau\in Z^s: \\ \tau_j>0, \lVert\tau\rVert<n}} \frac{1}{detA_{n,\tau}} \sum_{k\in K(n,\tau)} (R^2)^{-1}Rf\left(k\left(A_{n,\tau}^{-1}\right)'\right) \sum_{m\in\rho(\tau)} \int_{y\in R_x^{s-1}} exp\left(2\pi i\left(m, y-k\left(A_{n,\tau}^{-1}\right)'\right)\right) dy$
- Smolyak-type grids generated by the method of tensor products of functionals.
$\sum_{(\nu_1,\dots,\nu_s)\in\Omega\subset Z_{\nu_0}^s} \frac{1}{2^{\nu_1+\dots+\nu_s}} \sum_{k_1=0}^{2^{\nu_1}-1} \dots \sum_{k_s=0}^{2^{\nu_s}-1} f\left(\frac{k_1}{2^{\nu_1}},\dots,\frac{k_s}{2^{\nu_s}}\right) \left\{\prod_{j=1}^{s} \sum_{n_j=-2^{\nu_j}-1}^{2^{\nu_j}-1} \left[\lambda_{n_j}^{(\nu_j)}(j) - \right.\right.$ $\left.\left. - sgn\left(\nu_j-\nu_j^{(0)}\right)\left(1+(-1)^{k_j}\right)\lambda_{n_j}^{(\nu_j-1)}(j)\right] exp\left(2\pi i n_j\left(y_j-\frac{k_j}{2^{\nu_j}}\right)\right)\right\}.$
$\sum_{(\nu_1,\dots,\nu_s)\in\Omega\subset Z_{\nu_0}^s} \frac{1}{2^{\nu_1+\dots+\nu_s}} \sum_{k_1=0}^{2^{\nu_1}-1} \dots \sum_{k_s=0}^{2^{\nu_s}-1} (R^2)^{-1}Rf\left(\frac{k_1}{2^{\nu_1}},\dots,\frac{k_s}{2^{\nu_s}}\right) \int_{y\in R_x^{s-1}} \left\{\prod_{j=1}^{s} \sum_{n_j=-2^{\nu_j}-1}^{2^{\nu_j}-1} \left[\lambda_{n_j}^{(\nu_j)}(j) - \right.\right.$ $\left.\left. - sgn\left(\nu_j-\nu_j^{(0)}\right)\left(1+(-1)^{k_j}\right)\lambda_{n_j}^{(\nu_j-1)}(j)\right] exp\left(2\pi i n_j\left(y_j-\frac{k_j}{2^{\nu_j}}\right)\right)\right\} dy$