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A.V. Rezbaev

Department of Mathematics, National Research University Higher School of Economics, Usacheva str., 6, Moscow, 109028, Russian Federation
(E-mail: aratyo@yandex.ru)

On logarithmic derivatives of probability densities

Abstract: We construct two examples connected with the integrability of logarithmic derivatives of probability densities on the real line, in particular, with the Fisher information number. These examples show that the Fisher information of a probability density cannot be estimated in terms of L^1-norms of its first and second derivatives and the maximum of the absolute value of the second derivative. In addition, the norm of the logarithmic derivative of the density in L^3 cannot be estimated in terms of the norms in L^1 of the derivatives of the density of any order.

Keywords: Fisher information, logarithmitic derivative, Uglanov's lemma, Krugova's inequality

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1. Introduction. Let ρ be a probability density on the real line that is absolutely continuous on all bounded intervals. the ratio

$$\frac{\rho'(x)}{\rho(x)}$$

is the called the logarithmic derivative of ρ. We set $\rho'(x)/\rho(x) = 0$ if $\rho(x) = 0$. The expression

$$I(\rho) = \int_{\mathbb{R}} \frac{|\rho'(t)|^2}{\rho(t)} dt$$

is called the Fisher information number (or just the Fisher information) of density ρ. It is of interest for many applications to have efficient conditions on ρ under which this integral is finite. A.V. Uglanov [1] suggested the following elementary but not obvious lemma.

Lemma [U]. There exists a constant C such that for each nonnegative twice differentiable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that φ'' is absolutely continuous the following estimate holds:

$$J(\varphi) = \int_{\mathbb{R}} \frac{|\varphi'(t)|^2 dt}{\varphi(t)} \le C \int_{\mathbb{R}} [|\varphi'(t)| + |\varphi''(t)| + |\varphi'''(t)|] dt.$$

Later Krugova generalized this results as follows (see [2] and [3]):

Let recall the following know result

The Krugova inequality:

Let f be a nonnegative twice differentiable function on $\mathbb{R}$ such that f'' is absolutely continuous. Then for any $\varepsilon \in (0, 3)$ holds:

$$\int_{\mathbb{R}} \left(\frac{|f'(t)|}{f(t)} \right)^{3-\varepsilon} f(t) dt \le C(\varepsilon) \left(\int_{\mathbb{R}} |f'(t)| dt + \int_{\mathbb{R}} |f''(t)| dt + \int_{\mathbb{R}} |f'''(t)| dt \right).$$

In relation to Uglanov's lemma the question arises about the validity of this estimate if we require only the boundedness and integrability of the second derivative without the integrability of the third derivative. It turns out that the answer is negative. We are going go construct such a density in Theorem 1.

What happens if f is three times differentiable? Can one obtain similar estimates with $4-\varepsilon$? The answer is negative! Even analyticity is not enough, as the following simple example shows: for $\varphi(t)=t^2\exp(-t^2)$ the function $|\varphi'(t)|^p\varphi^{1-p}(t)$ is not integrable at the origin for all $p\geq 3$. In this example the density vanishes at the origin, but in Theorem 2 we construct a strictly positive density with the same properties. These examples answer two questions posed by S.G. Bobkov.

2. Main results.

Theorem 1. There exists a function $f\colon \mathbb{R}\to[0,\infty]$ such that

1) f, f', f'' are integrable on $\mathbb{R}$;

2) $|f''(x)|<M, \forall x\in\mathbb{R}$;

3) The Fisher information is infinite:

$$\int_{\mathbb{R}}\frac{|f'(x)|^2}{f(x)}dx=\infty.$$

Proof. Step 1. Let us construct such a function f. We the functions f_n defined on $\mathbb{R}$ by

$$f_n(x)=-\frac{x(x-1)}{n\ln^2 n}\chi_{[0,1]}(x),$$

where $n=4,6,8,\dots$.

One can readily verify the following simple properties of f_n:

(i) $0\leq f_n(x)dx\leq\dfrac{1}{4n\ln^2 n}$.

(ii) $\displaystyle\int_0^1|f_n'(x)|dx=\frac{1}{2n\ln^2 n}$.

(iii) $\int_0^1|f_n''(x)|dx=\frac{1}{n\ln^2 n}$.

(iv) $|f_n''(x)|\leq 1$ for all $n=4,6,8,\dots$.

Let us consider the following functions:

$$s_n(x)=y_{l,n}(x)+f_n'(x)\chi_{(\frac{1}{n},1-\frac{1}{n})}(x)+y_{r,n}(x),\ n=4,6,8,\dots,$$

where

$$y_{l,n}(x)=\frac{\left(1-\frac{2}{n}\right)^2}{2\left(1-\frac{1}{n}\right)\ln^2 n}\left(x+\frac{1}{n-2}\right)\chi_{[-\frac{1}{n-2},\frac{1}{n}]}(x),$$

$$y_{r,n}(x)=\frac{\left(1-\frac{2}{n}\right)^2}{2\left(1-\frac{1}{n}\right)\ln^2 n}\left(x-1-\frac{1}{n-2}\right)\chi_{[1-\frac{1}{n},1+\frac{1}{n-2}]}(x).$$

Loosely speaking, we removed the discontinuities of f_n' at the points 0 and 1 by adding $y_{l,n}$ and $y_{r,n}$, moreover, as we shall in Step 2, our modification s_n preserves some properties of f_n'.

Let us consider the following functions g_n, where $n=4,6,8,\dots$:

$$g_n(x)=\int_{-1}^{x}s_n(t)dt. \tag{1}$$

Using the symmetry of s_n, we can check that each function g_n vanishes outside the interval $\left[-\frac{1}{n-2},1+\frac{1}{n-2}\right]$, therefore, in (1) the integral can be taken over $\left[-\frac{1}{n-2},x\right]$.

Step 2. On the interval $\left[\frac{1}{n},1-\frac{1}{n}\right]$ the function g_n is equal to f_n. Indeed, let $x\in[\frac{1}{n},1-\frac{1}{n}]$. Then

$$g_n(x)=\int_{-\frac{1}{n-2}}^{x}s_n(t)dt=\int_{-\frac{1}{n-2}}^{\frac{1}{n}}s_n(t)dt+\int_{\frac{1}{n}}^{x}s_n(t)dt=\frac{1}{n^2\ln^2 n}\left(1-\frac{1}{n}\right)+\int_{\frac{1}{n}}^{x}f_n'(t)dt$$

$$=\frac{1}{n^2\ln^2 n}\left(1-\frac{1}{n}\right)+f_n(x)-f_n(\frac{1}{x})=f_n(x).$$

Step 3. It follows from our construction that

$$\int_{\mathbb{R}}|g_n(x)|dx\leq\frac{1}{4n\ln^2 n}\left(1+\frac{2}{n-2}\right).$$

Step 4. According to the definition of s_n we have the following inequalities:

$$\int_{\mathbb{R}} |g_n'(x)|dx = 2\int_{-\frac{1}{n-2}}^{\frac{1}{n}} y_{l,n}(x)dx + \int_{\frac{1}{n}}^{1-\frac{1}{n}} |f_n'(x)|dx = \frac{\left(1-\frac{2}{n}\right)^2}{2\left(1-\frac{1}{n}\right)\ln^2 n}\left(\frac{1}{n}+\frac{1}{n-2}\right)^2 + \int_{\frac{1}{n}}^{1-\frac{1}{n}} |f_n'(x)|dx,$$

$$\int_{\mathbb{R}} |g_n''(x)|dx = 2\int_{-\frac{1}{n-2}}^{\frac{1}{n}} y_{l,n}'(x)dx + \int_{\frac{1}{n}}^{1-\frac{1}{n}} |f_n''(x)|dx = \frac{\left(1-\frac{2}{n}\right)^2}{\left(1-\frac{1}{n}\right)\ln^2 n}\left(\frac{1}{n}+\frac{1}{n-2}\right) + \int_{\frac{1}{n}}^{1-\frac{1}{n}} |f_n''(x)|dx.$$

Step 5. Now we are going to prove the following assertion: the integral

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{|f_n'(x)|^2}{f_n(x)}dx$$

is of order as $\frac{1}{n\ln n}$ as $n\to\infty$. Using this assertion along with (ii), (iii) and the previous steps and taking into account that the series $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$ diverges and the series $\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}$ converges, we conclude that the function

$$f(x) = \sum_{n=2}^{\infty} g_{2n}(x-2n)$$

has the properties announced in Theorem 1. So it remains to prove the assertion above.

We have the following chain of equalities:

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{|f_n'(x)|^2}{f_n(x)}dx = -\frac{1}{n\ln^2 n}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{(2x-1)^2dx}{x(x-1)}$$

$$= -\frac{1}{n\ln^2 n}\left(\int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{4xdx}{x-1} - \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{4dx}{x-1} + \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{dx}{x(x-1)}\right)$$

$$= -\frac{1}{n\ln^2 n}\left(\int_{\frac{1}{n}}^{1-\frac{1}{n}} 4dx + \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{dx}{x(x-1)}\right)$$

$$= -\frac{1}{n\ln^2 n}\left(4\left(1-\frac{2}{n}\right) + \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{dx}{x-1} - \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{dx}{x}\right)$$

$$= -\frac{1}{n\ln^2 n}\left(4-\frac{8}{n}-2\left(\ln n+\ln\left(1-\frac{1}{n}\right)\right)\right) = \frac{2}{n\ln n} - \frac{1}{n\ln^2 n}\left(4-\frac{8}{n}-2\ln\left(1-\frac{1}{n}\right)\right).$$

This completes the proof of Theorem 1.

Our second example is this.

Theorem 2. There exists a function $f\colon \mathbb{R}\to(0,+\infty)$ with the following properties:

1) f is infinitely differentiable;

2) f is integrable on $\mathbb{R}$ along with all its derivatives;

3) $\displaystyle\int_{\mathbb{R}} \frac{|f'(x)|^3}{f^2(x)}dx = \infty$.

Proof. The main idea of this example is this. We observe that if we do not require that f be strictly positive, then, for example, the functions $\dfrac{\sin^2 x}{x^2}$ or $x^2e^{-x^2}$ are suitable, moreover, for the first function the expression

$$\frac{|f'(x)|^3}{f^2(x)} = \frac{8\sin^2 x}{x^5}(x\cot x - 1)^3$$

has non-integrable singularities at the points $x_n = \pi n$, where $n = \pm1, \pm2, \pm3, \dots$ (it is important here that they form a countable set, not finite). We are going to use it to construct the desired function. Let us consider the following positive function on $[\frac{1}{2}, +\infty)$:

$$g(x) = \frac{\sin^2 x + e^{-x}}{x^2}.$$

Since our function is not integrable in a neighborhood of zero, we have to extend it to the whole real line. Let us observe that

$$\int_1^\infty \frac{|g'(x)|^3}{g^2(x)}dx = \infty.$$

Indeed, it is easy to see that

$$g'(x) = \frac{2x\sin x\cos x - xe^{-x} - 2\sin^2 x - 2e^{-x}}{x^3}$$

and

$$\frac{|g'(x)|^3}{g^2(x)} = \frac{\left|2x\sin x\cos x - xe^{-x} - 2\sin^2 x - 2e^{-x}\right|^3}{x^5(\sin^2 x + e^{-x})^2}.$$

We notice that on the intervals $[\pi n - \frac{\pi}{2}, \pi n]$, where $n \geq 1$, all terms are non-positive, hence for such x we have

$$\frac{|g'(x)|^3}{g^2(x)} = \frac{\left(-2x\sin x\cos x + xe^{-x} + 2\sin^2 x + 2e^{-x}\right)^3}{x^5(\sin^2 x + e^{-x})^2} \geq -\frac{8x^3\sin^3 x\cos^3 x}{x^5(\sin^2 x + e^{-x})^2} = -\frac{8\sin^3 x\cos^3 x}{x^2(\sin^2 x + e^{-x})^2}.$$

Further, for the same x it holds

$$-\frac{8\sin^3 x\cos^3 x}{x^2(\sin^2 x + e^{-x})^2} \geq -\frac{8\sin^3 x\cos^3 x}{(\pi n)^2(\sin^2 x + e^{-(\pi n - \frac{\pi}{2})})^2}.$$

Now let $x \in [\pi n - \frac{\pi}{2}, \pi n - e^{-n}]$, where $n \geq 1$. Since $\sin x \neq 0$ for such x, we have

$$-\frac{8\sin^3 x\cos^3 x}{(\pi n)^2(\sin^2 x + e^{-(\pi n - \frac{\pi}{2})})^2} = -\frac{8\cos^3 x}{(\pi n)^2\left(1 + \frac{e^{-(\pi n - \frac{\pi}{2})}}{\sin^2 x}\right)^2 \sin x}$$

$$\geq -\frac{8\cos^3 x}{(\pi n)^2\left(1 + \frac{e^{-(\pi n - \frac{\pi}{2})}}{\sin^2(\pi n - e^{-n})}\right)^2 \sin x} = -\frac{8\cos^3 x}{(\pi n)^2\left(1 + \frac{e^{-(\pi n - \frac{\pi}{2})}}{\sin^2(e^{-n})}\right)^2 \sin x}.$$

Since

$$1 + \frac{e^{-(\pi n - \frac{\pi}{2})}}{\sin^2(e^{-n})} \to 1 \quad \text{as } n \to \infty,$$

we obtain $\left(1 + \frac{e^{-(\pi n - \frac{\pi}{2})}}{\sin^2(e^{-n})}\right)^2 \leq 2$ for large $n \geq N_0$.

Let us summarize our results. There exists $N_0 \in \mathbb{N}$ such that for all $x \in [\pi n - \frac{\pi}{2}, \pi n - e^{-n}]$ and $n \geq N_0$ it holds

$$\frac{|g'(x)|^3}{g^2(x)} \geq -\frac{4\cos^3 x}{(\pi n)^2 \sin x} \geq 0.$$

We observe that

$$\int_{\pi n - \frac{\pi}{2}}^{\pi n - e^{-n}} \frac{-\cos^3 x}{\sin x}dx = \int_{\frac{\pi}{2}}^{-e^{-n}} \frac{-\cos^3 x}{\sin x}dx = \int_{-1}^{-\sin(e^{-n})} \frac{-1 + x^2}{x}dx = -\ln\sin e^{-n} + \frac{\sin^2(e^{-n})}{2} - \frac{1}{2}.$$

Finally, since the series

$$\sum_{n=1}^{\infty} \frac{\ln\sin e^{-n}}{n^2}$$

diverges and both series

$$\sum_{n=1}^{\infty} \frac{\sin^2 e^{-n}}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge, we obtain

$$\int_1^\infty \frac{|g'(x)|^3}{g^2(x)}dx \geq \sum_{n=N_0}^{\infty} \int_{\pi n - \frac{\pi}{2}}^{\pi n - e^{-n}} -\frac{4\cos^3 x}{(\pi n)^2 \sin x}dx = \infty.$$

To complete the construction of f we need the following function:

$$\lambda(x) = \frac{\alpha(2-|x|)}{\alpha(2-|x|)+\alpha(|x|-1)},$$

where $\alpha(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{when} \quad x > 0 \\ 0, & \text{when} \quad x \le 0 \end{cases}$ Clearly, $\lambda \in C^\infty(\mathbb{R})$ and $\lambda(x) = 1$ when $x \in [-1,1]$, and $\lambda(x) = 0$ when $x \notin [-2,2]$. Finally, we reflect g evenly, and on the interval $(-\frac{1}{2}, \frac{1}{2})$ define it by zero, and let

$$f(x) = \alpha(x) + (1-\alpha(x))g(x).$$

This completes the proof of Theorem 2.

3. Conclusion. The density φ in Theorem 1 demonstrates that if we require only the boundedness of the second derivative without the integrability of the third derivative φ''', it can happen that the Fisher information of density φ can be infinite.

The density f in Theorem 2 gives an example of an analytic function without zeros with the following properties:

$$\int_{\mathbb{R}} \left(\frac{|f'(t)|}{f(t)}\right)^3 f(t)dt = \infty,$$

$$\int_{\mathbb{R}} |f^{(n)}(t)|dt < \infty$$

for all $n \ge 1$.

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А.В. Резбаев

Математика факультеті, Экономиканың жоғарғы мектебі ұлттық зерттеу университеті, Усачева көш., 6, Мәскеу, 109028, Ресей Федерациясы

Ықтималдық тығыздықтардың логарифмдік туындылары туралы

Аннотация: Ықтималдық тығыздықтардың логарифмдік туындыларының интегралдануымен, дәлірек айтқанда Фишер ақпараттық сандарымен байланысты екі мысал құрылды. Құрылған мысалдар ықтималдық тығыздықтың Фишер ақпараттық санын тығыдықтың бірінші және екінші туындыларының модулінің интегралы мен екінші туындының модульінің максимумы арқылы бағалауға болмайтындығы көрсетілді. Сонымен қатар, L^3 -тегі тығыздықтың логарифмдік туындысының нормасын тығыздықтың қандай да ретті туындысының L^1 -дегі нормасы арқылы бағалауға болмайтындығы көрсетілді.

Ключевые слова: Фишердің ақпараттық саны, логарифмдік туынды, Угланов леммасы, Круговая теңсіздгі.

А.В. Резбаев

Факультет математики, Национальный исследовательский университет Высшая Школа Экономики, ул. Усачева 6, Москва, 109028, Российская Федерация

О логарифмических производных вероятностных плотностей

Аннотация: Построены два примера, связанные с интегрируемость логарифмических производных вероятностных плотностей, в частности с информационным числом Фишера. Эти примеры показывают, что информационное число Фишера вероятностной плотности нельзя оценить через интегралы от модулей первой и второй производных плотности и максимум модуля второй производной. Кроме того, норму логарифмической производной плотности в L^3 нельзя оценить через нормы в L^1 производных плотности какого-либо порядка.

Ключевые слова: Информационное число Фишера, логарифмическая производная, лемма Угланова, неравенства Круговой

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Information about author:

Резбаев А.В. – Математика факультетінің екінші курс аспиранты, Экономиканың жоғарғы мектебі ұлттық зерттеу университеті,Усачева көш., 6, Мәскеу, 109028, Ресей Федерациясы.

Rezbaev A.V. – 2^{nd} year postgraduate student of mathematics department, Department of Mathematics, National Research University Higher School of Economics, Usacheva str., 6, Moscow, 109028, Russian Federation.