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**Cesàro means of negative order of the one-dimensional Vilenkin-Fourier series**

**Abstract:** In [1] has been proved some inequalities related to the approximation properties of Cesàro means of negative order of the one-dimensional Vilenkin-Fourier series. These inequalities allow one to obtain a sufficient condition for the convergence of Cesàro means of Vilenkin-Fourier series in the $L^p-$ metric in the term of modulus of continuity. In this paper, we will prove the sharpness of these conditions, in particular we find a continuous function under some condition of modulo of continuity, for which Cesàro means of Vilenkin-Fourier series diverge in the $L^p-$ metric.

**Keywords:** Inequalities, Approximation, Vilenkin system, Vilenkin-Fourier series, Cesàro means, Convergence in norm.

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## 1. Introduction

Let $N_+$ denote the set of positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, ...)$ denote a sequence of positive integers not less then 2. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo $m_k$. Define the group $G_m$ as the complete direct product of the groups $Z_{m_j}$, with the product of the discrete topologies of $Z_{mj}$ 's.

The direct product of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \qquad (j \in Z_{m_k})$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$. If the sequence $m$ is bounded, then $G_m$ is called a bounded Vilenkin group. In this paper we will consider only bounded Vilenkin groups. The elements of $G_m$ can be represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, $\left(x_j \in Z_{m_j}\right)$. The group operation $+$ in $G_m$ is given by

$$x + y = ((x_0 + y_0) \, mod \, m_0, ..., (x_k + y_k) \, mod \, m_k, ...),$$

where $x := (x_0, ..., x_k, ...)$ and $y := (y_0, ..., y_k, ...) \in G_m$. The inverse of $+$ will be denoted by $-$. For every $x \in G_m$ we denote $|x| := \sum\limits_{j=0}^{\infty} \frac{x_j}{M_{j+1}}, \left(x_j \in Z_{m_j}\right)$.

It is easy to give a base for the neighborhoods of $G_m$ :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m | y_0 = x_0, ..., y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $n \in N$. Define $I_n := I_n(0)$ for $n \in N_+$.

If we define the so-called generalized number system based on $m$ in the following way: $M_0 := 1,\ M_{k+1} := m_k M_k \ \ (k \in N)$, then every $n \in N$ can be uniquely expressed as $n = \sum\limits_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j} \ \ (j \in N_+)$ and only a finite number of $n_j$'s differ from zero. We also use the following notation: $|n| := \max\{k \in N : n_k \neq 0\}$ (that is , $M_{|n|} \leq n < M_{|n|+1}$ ).

Next, we introduce $G_m$ on an orthonormal system, which is called Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to C$, the generalized Rademacher functions, in this way:

$$r_k(x) := \exp \frac{2\pi i x_k}{m_k} \quad \left(i^2 = -1,\ x \in G_m,\ k \in N\right).$$

Now we define the Vilenkin system $\psi := (\psi_n : n \in N)$ on $G_m$ as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad (n \in N).$$

In particular, we call the system the Walsh-Paley system if $m = 2$.

The Vilenkin system is orthonormal and complete in $L^1(G_m)$ ( see [2]).

Now, introduce analogues of the usual definitions of the Fourier analysis. If $f \in L^1(G_m)$ we can establish the following definitions in the usual way:

Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi_k} d\mu, \qquad (k \in N),$$

partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k, \quad (n \in N_+,\ S_0 f := 0),$$

Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k, \qquad (n \in N_+).$$

The $(C, -\alpha)$ means of the Vilenkin-Fourier series are defined as

$$\sigma_n^{-\alpha}(f, x) = \frac{1}{A_n^{-\alpha}} \sum_{k=0}^{n} A_{n-k}^{-\alpha} \widehat{f}(k)\,\psi_k(x),$$

where

$$A_0^{\alpha} = 1, \qquad A_n^{\alpha} = \frac{(\alpha+1)\ldots(\alpha+n)}{n!}.$$

It is well Known that [3]

$$A_n^{\alpha} = \sum_{k=0}^{n} A_k^{\alpha-1}.$$

$$A_n^{\alpha} - A_{n-1}^{\alpha} = A_n^{\alpha-1}.$$

$$A_n^{\alpha} \sim n^{\alpha}.$$

The norm of the space $L^p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p\, d\mu(x)\right)^{1/p}, \quad (1 \leq p < \infty).$$

Denote by $C(G_m)$ the class of continuous functions on the group $G_m$, endoved with the supremum norm.

For the sake of brevity in notation, we agree to write $L^{\infty}(G_m)$ instead of $C(G_m)$.

Let $f \in L^p(G_m), 1 \leq p \leq \infty$. The expression

$$\omega\left(\frac{1}{M_n}, f\right)_p = \sup_{h \in I_n} \|f(\cdot - h) - f(\cdot)\|_p$$

is called the modulus of continuity.

The problems of summability of partial sums and Cesàro means for Walsh-Fourier series were studied in [4], [5]- [12], [13]. In his monography [14] Zhizhinashvili investigated the behavior of Cesàro method of negative order for trigonometric Fourier series in detail. Goginava [5] studied analogical question in case fo the Walsh system. The analogous results in the case of the Vilenkin-Fourier series have been studied in [1]. In particular, the following was proved:

**Theorem T.** *[1] Let $f$ belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and $\alpha \in (0,1)$. Then for any $M_k \leq n < M_{k+1}$ $(k, n \in N)$ the inequality*

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \leq c(p,\alpha)\left\{M_k^{\alpha}\omega(1/M_{k-1}, f)_p + \sum_{r=0}^{k-2}\frac{M_r}{M_k}\omega(1/M_r, f)_p\right\}$$

*holds true.*

This result allows one to obtain the condition which is sufficient for the convergence of the means $\sigma_n^{-\alpha}(f,x)$ to $f(x)$ in the $L^p-$ metric.

**Corollary 1.** [1] Let $f$ belong to $L^p(G_m)$ for some $p \in [1,\infty]$ and let $\alpha \in (0,1)$. If

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = o\left(\frac{1}{M_k^{\alpha}}\right),$$

then

$$\left\|\sigma_n^{-\alpha}(f) - f\right\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

In this paper, we are going to prove the sharpness of Corollary 1. In particular, the following Theorem holds:

**Theorem 1.** *For every $\alpha \in (0,1)$, there exists a function $f \in C(G_m)$ for which*

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_C = O\left(\frac{1}{M_k^{\alpha}}\right),$$

*and*

$$\limsup_{k\to\infty}\left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_1 > 0.$$

Since for a continuous function we have proved divergence in the space $L_1$, we can conclude the following corollary:

**Corollary 2.** For every $\alpha \in (0,1)$, there exists a function $f \in C(G_m)$, for which

$$\omega\left(f, \frac{1}{M_{k-1}}\right)_p = O\left(\frac{1}{M_k^{\alpha}}\right),$$

and

$$\limsup_{k\to\infty}\left\|\sigma_{M_k}^{-\alpha}(f) - f\right\|_p > 0, \quad \text{for some } p \in [1,\infty].$$

## 2. Proofs of the main results

Proof of Theorem 1.

We define the function

$$f(x)=\sum_{j=1}^{\infty}\frac{1}{M_j^{\alpha}}f_j(x),$$

where

$$f_j(x)=\rho_j(x)=\exp\frac{2\pi i x_j}{m_j}.$$

First, we prove that

$$\omega\left(f,\frac{1}{M_n}\right)_C=O\left(\frac{1}{M_n^{\alpha}}\right). \tag{1}$$

Since

$$|f_j(x-t)-f_j(x)|=0,\qquad j=0,1,...,n-1,\qquad t\in I_n$$

we get

$$|f(x-t)-f(x)|\leq\sum_{j=1}^{n-1}\frac{1}{M_j^{\alpha}}|f_j(x-t)-f_j(x)|$$

$$+\sum_{j=n}^{\infty}\frac{2}{M_j^{\alpha}}\leq\frac{c}{M_n^{\alpha}}.$$

After we showed that 1 holds, next, we shall prove that $\sigma_{M_k}^{-\alpha}(f)$ diverges in the $L^1$ metric. It is clear that

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1\geq\left|\int_{G_m}\left[\sigma_{M_k}^{-\alpha}(f,x)-f(x)\right]\psi_{M_k}(x)\,d\mu(x)\right| \tag{2}$$

$$\geq\left|\int_{G_m}\sigma_{M_k}^{-\alpha}(f,x)\,\psi_{M_k}(x)\,d\mu(x)\right|-\left|\widehat{f}(M_k)\right|$$

$$=\left|\frac{1}{A_{M_k}^{-\alpha}}\sum_{i=0}^{M_k}A_{M_k-i}^{-\alpha}\widehat{f}(i)\int_{G_m}\psi_i(x)\,\psi_{M_k}(x)\,d\mu(x)\right|$$

$$-\left|\widehat{f}(M_k)\right|=\frac{1}{A_{M_k}^{-\alpha}}\left|\widehat{f}(M_k)\right|-\left|\widehat{f}(M_k)\right|.$$

We have

$$\widehat{f}(M_k)=\int_{G_m}f(x)\,\bar{\psi}_{M_k}(x)\,d\mu(x)$$

$$=\sum_{j=1}^{\infty}\frac{1}{M_j^{\alpha}}\int_{G_m}\rho_j(x)\,\bar{\psi}_{M_k}(x)\,d\mu(x)=\frac{1}{M_k^{\alpha}}.$$

So, we can write

$$\left\|\sigma_{M_k}^{-\alpha}(f)-f\right\|_1\geq c(\alpha). \tag{3}$$

Theorem 1 is proved.

## 3. Conclusion

Theorem 1 gives the $L^p$ norm estimation of the difference between Cesàro means of negative order of the one-dimensional Vilenkin-Fourier Series and functions from $L^p$. This inequality allows one to obtain a sufficient condition for the convergence of the Cesàro means to $f(x)$ in the $L^p-$ metric, as discussed in Corollary 1. In this paper we proved the sharpness of Corollary 1. In particular, for a continuous function, we showed divergence in the space $L_1$, from which follows divergence in the space $L_p$, with $p \in [1,\infty]$.

## References


1 Tepnadze T. On the approximation properties of CesÃ ro means of negative order of Vilenkin-Fourier series // Studia Sci. Math. Hung. –2016. –Vol.53, №4. –P.532-544.

2 Agaev G.N., Vilenkin N.Ya., Dzhafarli G.M., and Rubinshtejn A.I. Multiplicative systems of functions and harmonic analysis on zero-dimensional groups. –Baku: Ehlm, 1981 [in Russian].

3 Zygmund A. Trigonometric series. Vo1.1. –Cambridge: Cambridge University Press, 1959.

4 Fine N.J. Cesàro summability of Walsh-Fourier series // Proc. Nat.Acad. Sci. U.S.A. –1995. –Vol. 41. –P. 558-591.

5 Goginava U. On the approximation properties of Cesàro means of negative order of Walsh-Fourier series // J. Approx. Theory. –2002. –Vol.115, №1. –P.9-20.

6 Goginava U. Uniform convergence of Cesàro means of negative order of double Walsh-Fourier series // J. Approx. Theory 2003. –Vol.124, №1. –P 96-108.

7 Goginava U., Nagy K. On the maximal operator of Walsh-Kaczmarz-Fejér means // Czechoslovak Math. J. –2011. –Vol. 61(136), №3. –P. 673-686.

8 Gát G., Goginava U. A weak type inequality for the maximal operator of $(C,\alpha)$-means of Fourier series with respect to the Walsh-Kaczmarz system // Acta Math. Hungar. –2009. –Vol. 125, №1-2. –P. 65-83.

9 Gát G. and Nagy K., Cesàro summability of the character system of the p-series field in the Kaczmarz rearrangement // Anal. Math. –2002. –Vol. 28, №1. –P. 1-23.

10 Nagy K. Approximation by Cesàro means of negative order of Walsh-Kaczmarz-Fourier series // East J. Approx. –2010. –Vol.16, №3. –P. 297-311.

11 Simon P., Weisz F. Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series // J. Approx. Theory. –2008. –Vol. 151, №1. –P. 1-19.

12 F. Schipp, Über gewisse Maximaloperatoren // Ann. Univ. Sci. Budapest. Sect. Math. –1975. –Vol. 18. –P. 189-195.

13 Tevzadze V. I. Uniform $(C,-\alpha)$ summability of Fourier series with respect to the Walsh-Paley system // Acta Math. Acad. Paedagog. Nyházi. (N.S.). –2006. –Vol. 22№1ю –P. 41–61 (electronic).

14 Zhizhiashvili L.V. Trigonometric Fourier series and their conjugates. Tbilisi, 1993 (Russian); English transl.: Kluwer Acad. publ, 1996.

15 Golubov B. I., Efimov A.V., and Skvortsov V.A. Series and transformation of Walsh. Moscow: Nauka, 1987 [In Russian]; English translation, Kluwer Academic, Dordrecht, 1991.

16 Goginava U. On the uniform convergence of Walsh-Fourier series // Acta Math. Hungar. –2001. –Vol. 93, №1-2. –P. 59-70.

17 Schipp F., Wade W.R., Simon P. and Pál J., Walsh Series, Introduction to Dyadic Harmonic Analysis. Hilger, Bristol, 1990.


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**Бір өлшемді Виленкин-Фурье қатарының теріс ретті Чезàро орташалары**

**Аннотация:** [1]-де бір өлшемді Виленкин-Фурье қатарларының теріс ретті Чезàро орташаларының жуықтау қасиеттерімен байланысты кейбір теңсіздіктер дәлелденген. Бұл теңсіздіктер $L^p$ - метрикасында үзіліссіздік модульдері терминдерінде Виленкин - Фурье қатарының Чезàро орташаларының жинақталуының жеткілікті шартын алуға мүмкіндік береді. Бұл мақалада біз осы шарттың дәл екенін көрсетеміз, дербес жағдайда Виленкин-Фурье қатарының Чезаро орталары $L^p$ метрикасында жинақталмайтындай үзіліссіздік модулі белгілі бір шарттарды қанағаттандыратын үзіліссіз функция құрылады.

**Түйін сөздер:** Теңсіздіктер, жуықтау, Виленкин жүйесі, Виленкин-Фурье қатары, Чезàро орташаалары , норма бойынша жинақтылық.

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**Средние Чезàро отрицательного порядка одномерного ряда Виленкина-Фурье**

**Аннотация:** В [1] доказаны некоторые неравенства, связанные с аппроксимационными свойствами средних Чезàро с отрицательным порядком одномерных рядов Виленкина-Фурье. Эти неравенства позволяют получить достаточное условие сходимости средних Чезàро рядов Виленкина – Фурье в $L^p$-метрике в терминах модуля непрерывности. В данной статье мы докажем точность этого условия, в частности найдена непрерывная функция с некоторыми условия на ее модуль непрерывности, для которой средние Чезаро рядов Виленкина-Фурье расходятся в метрике $L^p$ .

**Ключевые слова:** Неравенства, аппроксимация, система Виленкина, ряд Виленкина-Фурье, средние Чезàро, сходимость по норме.

## References

1 Tepnadze T. On the approximation properties of CesÃ ro means of negative order of Vilenkin-Fourier series. Studia Sci. Math. Hung. 53(4), 532-544 (2016).
2 Agaev G.N., Vilenkin N.Ya., Dzhafarli G.M., and Rubinshtejn A.I. Multiplicative systems of functions and harmonic analysis on zero-dimensional groups (Baku, Ehlm, 1981) [in Russian].
3 Zygmund A. Trigonometric series. Vo1.1. (Cambridge, Cambridge University Press, 1959).
4 Fine N.J. Cesàro summability of Walsh-Fourier series. Proc. Nat.Acad. Sci. U.S.A. 41, 558-591 (1995).
5 Goginava U. On the approximation properties of Cesàro means of negative order of Walsh-Fourier series. J. Approx. Theory. 115(1), 9-20 (2002).
6 Goginava U. Uniform convergence of Cesàro means of negative order of double Walsh-Fourier series. J. Approx. Theory. 124(1), 96-108 (2003).
7 Goginava U., Nagy K. On the maximal operator of Walsh-Kaczmarz-Fejér means. Czechoslovak Math. J. 61(136), 3, 673-686 (2011).
8 Gát G., Goginava U. A weak type inequality for the maximal operator of $(C,\alpha)$ -means of Fourier series with respect to the Walsh-Kaczmarz system. Acta Math. Hungar. 125(1-2), 65-83 (2009).
9 Gát G. and Nagy K., Cesàro summability of the character system of the p-series field in the Kaczmarz rearrangement. Anal. Math. 28(1), 1-23 (2002).
10 Nagy K. Approximation by Cesàro means of negative order of Walsh-Kaczmarz-Fourier series. East J. Approx. 16(3), 297-311 (2010).
11 Simon P., Weisz F. Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series. J. Approx. Theory. 151(1), 1-19 (2008).
12 F. Schipp, Über gewisse Maximaloperatoren. Ann. Univ. Sci. Budapest. Sect. Math. 18, 189-195 (1975).
13 Tevzadze V. I. Uniform $(C,-\alpha)$ summability of Fourier series with respect to the Walsh-Paley system. Acta Math. Acad. Paedagog. Nyházi. (N.S.). 22(1), 41–61 (2006) (electronic).
14 Zhizhiashvili L.V. Trigonometric Fourier series and their conjugates. (Tbilisi, 1993 [in Russian], English transl.: Kluwer Acad. publ, 1996).
15 Golubov B. I., Efimov A.V., and Skvortsov V.A. Series and transformation of Walsh. (Moscow, Nauka, 1987 [In Russian], English translation, Kluwer Academic, Dordrecht, 1991).
16 Goginava U. On the uniform convergence of Walsh-Fourier series. Acta Math. Hungar. 93(1-2), 59-70 (2001).
17 Schipp F., Wade W.R., Simon P. and Pál J., Walsh Series, Introduction to Dyadic Harmonic Analysis. (Hilger, Bristol, 1990).

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