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ON h-HOLOMORPHY AND h-ANALYTICITY OF FUNCTIONS OF AN h-COMPLEX VARIABLE

Abstract: Interest in the study of the properties of functions defined on the set of h-complex numbers arose again in connection with existing applications in geometry and mechanics. In this paper, we present necessary and sufficient conditions for h-differentiability and h-holomorphy of functions of an h-complex variable, the theorem on finite increments is proved, sufficient conditions for h-analyticity are found, a uniqueness theorem for h-analytic functions is proved.

Keywords: h-differentiability, h-holomorphy, h-analyticity, Ring of h-complex numbers, Zero divisors, Zeros of a function

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Introduction

Let $\mathbb{C}_h$ be the set of all h-complex numbers [1-6], i.e. the set of ordered pairs of real numbers, on which the addition and multiplication operations are given according to the rules:

$$\forall z_1 = (a;b)\,, z_2 = (c;d) \in \mathbb{C}_h$$

1. $z_1 + z_2 = (a+c; b+d)\,;$
2. $z_1 \cdot z_2 = (ac+bd; ad+bc)\,.$

The real unit is identified by the h-complex number $(1;0)$. A hyperbolic unit is the h-complex number $j = (0;1)$. Then any number from $\mathbb{C}_h$ can be represented in algebraic form:

$$z = (a;b) = (a;0) + (0;b) = a \cdot (1;0) + b \cdot (0;1) = a + jb = \text{Re } z + j \text{ Hyp } z,$$

where $a =$ Re z is the real part of z, $b =$ Hyp z is the hyperbolic part of z.

As shown in [7,8], the set of h-complex numbers $\mathbb{C}_h$ is a zero-divisor ring. Zero-divisors are numbers of the form $a \pm aj$. Special mention should be the case when $a = \frac{1}{2}$, then zero divisors have the following properties:

(a) $\forall\, n \in \mathrm{N}: \ \left(\frac{1\pm j}{2}\right)^n = \frac{1\pm j}{2}\,;$

(b) the numbers $\frac{1\pm j}{2}$ form a basis in $\mathbb{C}_h$, i.e. any h-complex number $a+bj$ can be uniquely represented as:

$$a + jb = (a+b)\,\frac{1+j}{2} + (a-b)\,\frac{1-j}{2}.$$

The norm of the element $z = a + jb$ in the ring $\mathbb{C}_h$ is defined as: $\|z\| = |a| + |b|$, and the modulus of a h-complex number is $|z| = \sqrt{a^2+b^2}$ as usual.

Let us present the properties of the norm:

1. $\|z\| = 0 \ \Leftrightarrow z = 0;$
2. $|z| = \sqrt{a^2+b^2} \le |a| + |b| = \|z\| \le \sqrt{2}\sqrt{a^2+b^2} = \sqrt{2}\,|z|\,;$
3. $\forall\, \alpha \in \mathbb{R}: \ \|\alpha z\| = |\alpha| \cdot \|z\|\,;$
4. $\forall\, z_1, z_2 \in \mathbb{C}_h: \ \|z_1 \cdot z_2\| \le \|z_1\| \cdot \|z_2\|\,;$
5. $\|z^n\| = \|z\|^n\,, \quad \forall n \in \mathbb{N}\,;$
6. $\frac{1}{\|z\|} \le \left\|\frac{1}{z}\right\|.$

On the set $\mathbb{C}_h$, the topology is introduced using the above norm.

h-complex argument functions

Let D be a domain in $\mathbb{C}_h$ and $f : D \to \mathbb{C}_h$.

Definition 1. The function f is called h-differentiable at the point $z \in D$ if there exists a number $k \in \mathbb{C}_h$ such that

$$f(z+h) - f(z) = kh + \alpha(h) \cdot h, \tag{1}$$

where $h \in D$ is not a divisor of zero, and $z + h \in D$, moreover, $\lim\limits_{h\to 0} \alpha(h) = 0$, k does not depend on h .

Definition 2. The derivative of a function f of an h-complex argument $z \in D$ is called

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

where $h \in \mathbb{C}_h$ is not a zero divisor. The limit is taken according to the norm from $\mathbb{C}_h$.

The derivative of the sum, difference, product, quotient of division, and composition of functions is calculated using the same formulas as in classical analysis .

Theorem 1. *The function f is h-differentiable at the point $z \in D$ if and only if there exists*

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

The proof is carried out in the same way as in the case of an analytic function of a complex variable in this case $f'(z) = k$ from (1).

Any h-complex function $f(z) = f(x + jy)$ is representable in algebraic form:

$$f(z) = u(x, y) + jv(x, y).$$

Theorem 2. *Let $f(z) = u(x, y) + jv(x, y)$ be defined in a neighborhood of the point $z = x + jy$, functions $u(x, y)$, $v(x, y)$ are differentiable at the point (x, y). Then two statements are equivalent:*

1) *function $f(z)$ h-differentiable at the point z;*
2) *the following equalities are true:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2}$$

P r o o f. Let us show that the 2) follows from the 1). Let be $h = s + jt$ and

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

We put $t = 0$:

$$f'(z) = \lim_{s\to 0} \frac{u(x+s, y) - u(x, y)}{s} + j \lim_{s\to 0} \frac{v(x+s, y) - v(x, y)}{s} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}.$$

Let be $s = 0$ then:

$$f'(z) = \lim_{t\to 0} \frac{u(x, y+t) - u(x, y)}{jt} + j \lim_{t\to 0} \frac{v(x, y+t) - v(x, y)}{jt} = j\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

We have:

$$\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + j\frac{\partial u}{\partial y}$$

consequently equalities (2) are true.

Now we show that 2) implies 1). Let the equality (2) be true then

$$f(z+h) - f(z) = [u(x+s, y+t) - u(x, y)] + j[v(x+s, y+t) - v(x, y)] =$$

$$= \left(u'_x s + u'_y t + \alpha(h) h\right) + j\left(v'_x s + v'_y t + \beta(h) h\right) = u'_x (s + jt) + jv'_x (s + jt) + (\alpha(h) + j\beta(h)) h =$$

$$= \left(u'_x + jv'_x\right) h + \gamma(h) h,$$

where $\gamma(h) = \alpha(h) + j\beta(h)$, $\lim\limits_{h\to 0}\gamma(h) = 0$. Consequently function $f(z)$ h-differentiable and

$$f'(z) = u_x^{'} + jv_x^{'} = v_y^{'} + ju_y^{'}.$$

The theorem is proved.

Remark 1. The equalities (2) are analogous to the Cauchy-Riemann conditions.

General form of h-holomorphic functions

Let the function f be h-differentiable in domain D.

Definition 3. The function $f(z) = u(x,y) + jv(x,y)$ is called h-holomorphic at the point $z_0 = x_0 + jy_0 \in D$ if the functions u and v have continuous second partial derivatives, and the conditions (2) are true.

Theorem 3. *The function* f *is* h*-holomorphic at the point* $z \in D$ *if and only if*

$$f(z) = \frac{1+j}{2} f(x+y) + \frac{1-j}{2} f(x-y). \tag{3}$$

P r o o f. Consider the function $f(z) = u(x,y) + jv(x,y)$. Let the condition (2) is true then the functions u and v satisfy the equations

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0; \qquad \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0. \tag{4}$$

Let $\xi = \frac{1}{2}(x+y)$, $\eta = \frac{1}{2}(x-y)$ and then

$$\begin{cases} \frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} x_\xi^{'} + \frac{\partial u}{\partial y} y_\xi^{'} = u_x^{'} + u_y^{'}, \\ \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} x_\eta^{'} + \frac{\partial u}{\partial y} y_\eta^{'} = u_x^{'} - u_y^{'}. \end{cases}$$

Mixed derivatives of functions u and v equal to zero

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0, \qquad \frac{\partial^2 u}{\partial \eta \partial \xi} = 0.$$

Thus, the equations (4) are equivalent to the following:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \eta \partial \xi} = 0. \tag{5}$$

Similarly, we obtain equations for the function v:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{\partial^2 v}{\partial \eta \partial \xi} = 0. \tag{6}$$

Let's find a general solution (5) and (6):

$$u_\xi^{'} = \mu^*(\xi)$$

$$u(\xi,\eta) = \int \mu^*(\xi)\, d\xi = \tilde{\mu}(\xi) + \tilde{\psi}(\eta) = \tilde{\mu}\left(\frac{x+y}{2}\right) + \tilde{\psi}\left(\frac{x-y}{2}\right) = \frac{1}{2}\left\{\mu(x+y) + \psi(x-y)\right\},$$

$$v_\xi^{'} = \phi^*(\xi)$$

$$v(\xi,\eta) = \int \phi^*(\xi)\, d\xi = \tilde{\phi}(\xi) + \tilde{\nu}(\eta) = \tilde{\varphi}\left(\frac{x+y}{2}\right) + \tilde{\nu}\left(\frac{x-y}{2}\right) = \frac{1}{2}\left\{\phi(x+y) + \nu(x-y)\right\}.$$

From equations

$$u_x^{'} = v_y^{'}, \qquad v_x^{'} = u_y^{'}$$

follows

$$\begin{cases} \frac{1}{2}\left\{\mu'(x+y) + \psi'(x-y)\right\} = \frac{1}{2}\left\{\phi'(x+y) - \nu'(x-y)\right\}, \\ \frac{1}{2}\left\{\mu'(x+y) - \psi'(x-y)\right\} = \frac{1}{2}\left\{\phi'(x+y) + \nu'(x-y)\right\}. \end{cases}$$

Consequently

$$\begin{cases} \mu'(x+y) = \phi'(x+y), \\ \psi'(x-y) = \nu'(x-y), \end{cases} \qquad \begin{cases} \mu(x+y) = \phi(x+y) + \alpha, \\ \psi(x-y) = \nu(x-y) + \beta, \end{cases}$$

$$\begin{cases} u(x,y) = \frac{1}{2}\{\phi(x+y) + \psi(x-y) + \alpha\}, \\ v(x,y) = \frac{1}{2}\{\phi(x+y) - \psi(x-y) + \beta\}. \end{cases}$$

We have:

$$\begin{cases} f(z) = f(x+jy) = u(x,y) + jv(x,y), \\ \overline{f}(z) = \overline{f}(x+jy) = u(x,y) - jv(x,y), \end{cases} \begin{cases} f(x) = u(x,0) + jv(x,0), \\ \overline{f}(x) = u(x,0) - jv(x,0). \end{cases}$$

Therefore,

$$\begin{cases} u(x,0) = \frac{1}{2}\left\{f(x) + \overline{f}(x)\right\}, \\ v(x,0) = \frac{1}{2}\left\{f(x) - \overline{f}(x)\right\}, \end{cases} \qquad \begin{cases} u(x,0) = \frac{1}{2}\{\phi(x) + \psi(x) + \alpha\}, \\ v(x,0) = \frac{1}{2}\{\phi(x) - \psi(x) + \beta\}, \end{cases}$$

which means

$$\begin{cases} \phi(x) = u(x,0) + v(x,0) - \left(\frac{\alpha+\beta}{2}\right), \\ \psi(x) = u(x,0) - v(x,0) - \left(\frac{\alpha-\beta}{2}\right), \end{cases}$$

then

$$\begin{cases} \phi(x) = \frac{1}{2}\left(f(x) + \overline{f}(x)\right) + \frac{j}{2}\left(f(x) - \overline{f}(x)\right) - \left(\frac{\alpha+\beta}{2}\right), \\ \psi(x) = \frac{1}{2}\left(f(x) + \overline{f}(x)\right) - \frac{j}{2}\left(f(x) - \overline{f}(x)\right) - \left(\frac{\alpha-\beta}{2}\right), \end{cases}$$

$$\begin{cases} \phi(x) = \frac{1+j}{2}f(x) + \frac{1-j}{2}\overline{f}(x) - \left(\frac{\alpha+\beta}{2}\right), \\ \psi(x) = \frac{1-j}{2}f(x) + \frac{1+j}{2}\overline{f}(x) - \left(\frac{\alpha-\beta}{2}\right). \end{cases}$$

From this we find that

$$f(z) = u(x,y) + jv(x,y) = \frac{1}{2}\{\phi(x+y) + \psi(x-y) + \alpha\} + \frac{j}{2}\{\phi(x+y) - \psi(x-y) + \beta\} =$$

$$= \frac{1}{2}\left\{\frac{1+j}{2}f(x+y) + \frac{1-j}{2}\overline{f}(x+y) - \left(\frac{\alpha+\beta}{2}\right) + \alpha + \frac{1+j}{2}f(x-y) + \frac{1-j}{2}\overline{f}(x-y) - \left(\frac{\alpha-\beta}{2}\right)\right\} +$$

$$+\frac{j}{2}\left\{\frac{1+j}{2}f(x+y) + \frac{1-j}{2}\overline{f}(x+y) - \left(\frac{\alpha+\beta}{2}\right) - \frac{1+j}{2}f(x-y) - \frac{1-j}{2}\overline{f}(x-y) + \left(\frac{\alpha-\beta}{2}\right) + \beta\right\} =$$

$$= \frac{1+j}{4}f(x+y) + \frac{1-j}{4}\overline{f}(x+y) + \frac{1-j}{4}f(x-y) + \frac{1+j}{4}\overline{f}(x-y) + \frac{1+j}{4}f(x+y) -$$

$$-\frac{1-j}{4}\overline{f}(x+y) + \frac{1-j}{4}f(x-y) - \frac{1+j}{4}\overline{f}(x-y) = \frac{1+j}{2}f(x+y) + \frac{1-j}{2}f(x-y).$$

Thus, the equality (3) is true.

Conversely, let (3)be true then for the function $f(z) = u(x,y) + jv(x,y)$, we put $y = 0$:

$$f(x) = u(x,0) + jv(x,0)$$

and then

$$\begin{cases} f(x+y) = u(x+y,0) + jv(x+y,0), \\ f(x-y) = u(x-y,0) + jv(x-y,0). \end{cases}$$

Using the equality (3) we represent the function $f(z)$ as:

$$f(z) = \frac{1+j}{2}\left[u(x+y,0) + jv(x+y,0)\right] + \frac{1-j}{2}\left[u(x-y,0) + jv(x-y,0)\right] =$$

$$= \frac{1}{2}\left[u(x+y,0) + v(x+y,0) + u(x-y,0) - v(x-y,0)\right] +$$

$$+\frac{j}{2}\left[u(x+y,0) + v(x+y,0) - u(x-y,0) + v(x-y,0)\right] = u(x,y) + jv(x,y).$$

Due to the fact that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x},$$

we get the proved theorem.

Theorem 4. *The function f is h-holomorphic in $D \subset \mathbb{C}_h$ with piecewise smooth boundary ∂D and is continuous in closure $\bar{D} = D \cup \partial D$. Then*

$$\int\limits_{\partial D} f(z)\, dz = 0.$$

P r o o f.

$$\int\limits_{\partial D} f(z)\, dz = \int\limits_{\partial D} [u(x,y) + jv(x,y)]\,(dx + jdy) = \int\limits_{\partial D} u(x,y)\,dx + v(x,y)\,dy + j\int\limits_{\partial D} u(x,y)\,dy + v(x,y)\,dx.$$

Using Green's formula we obtain

$$\int\limits_{\partial D} f(z)\, dz = \iint\limits_{D} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + j\iint\limits_{D} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0.$$

The theorem is proved.

Further, we need the following theorem of real analysis, which can be deduced from the second theorem "on finite increments" [9].

Theorem 5 (on finite increments for mappings from $\mathbb{R}^2$ into $\mathbb{R}^2$). *Let $F : \tilde{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be h-differentiable at the point $(a,b) \in \tilde{D}$. Then*

$$|F(a+s,\ b+t) - F(a,b)| \le \max_{\xi\in[0,1]} \left| F'(a+\xi s, b+\xi t) \begin{bmatrix} s \\ t \end{bmatrix} \right|. \tag{7}$$

P r o o f. We introduce an auxiliary function

$$g(\tau) = F(a+\tau s,\ b+\tau t), \qquad \tau \in [0,1].$$

We have:

$$g : [0,1] \to \mathbb{R}^2, \quad g(0) = F(a,\ b), \quad g(1) = F(a+s,\ b+t),$$
$$g' = F'(a+\tau s,\ b+\tau t) \begin{bmatrix} s \\ t \end{bmatrix}.$$

We put

$$G(\tau) = \langle g(t)\,|g(1) - g(0)\rangle,$$
$$G : [0,1] \to \mathbb{R}, \quad G'(\tau) = \langle g'(t)\,|g(1) - g(0)\rangle.$$

Hence, by Lagrange's theorem, it follows that

$$G(1) - G(0) = G'(\xi)\cdot 1, \quad \text{where} \quad \xi \in [0,1].$$

Using Cauchy's inequality for the scalar product we obtain

$$G(1) - G(0) = \langle g(1)\,|g(1) - g(0)\rangle - \langle g(0)\,|g(1) - g(0)\rangle = \langle g(1) - g(0)\,|g(1) - g(0)\rangle =$$
$$= |g(1) - g(0)|^2 = \langle g'(\xi)\,|g(1) - g(0)\rangle \le \langle |g'(\xi)|\ |\ g(1) - g(0)\rangle.$$

Consequently

$$|g(1) - g(0)| \le \max_{\xi\in[0,1]} |g'(\xi)|.$$

This inequality is equivalent to the following

$$|F(a+s,\ b+t) - F(a,b)| \le \max_{\xi\in[0,1]} \left| F'(a+\xi s, b+\xi t) \begin{bmatrix} s \\ t \end{bmatrix} \right|.$$

The theorem is proved.

Remark 2. We represent the function $F(x,y)$ in vector form $F(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ then

$$F'(x,y) = \begin{bmatrix} u'_x(x,y) & u'_y(x,y) \\ v'_x(x,y) & v'_y(x,y) \end{bmatrix}.$$

Now from the inequality (7) and the Cauchy-Bunyakovsky inequality we obtain

$$|F(a+s,\ b+t)-F(a,b)| = \left\| \begin{bmatrix} u(a+s,\ b+t)-u(a,b) \\ v(a+s,\ b+t)-v(a,b) \end{bmatrix} \right\| = \left\| \begin{bmatrix} \triangle u \\ \triangle v \end{bmatrix} \right\| = \sqrt{|\triangle u|^2+|\triangle v|^2} \le$$
$$\le \max_{\xi\in[0,1]} \sqrt{\left\{u'_x s+u'_y t\right\}^2+\left\{v'_x s+v'_y t\right\}^2} \le \max_{\xi\in[0,1]} \sqrt{\left\{\left|u'_x\right|^2+\left|u'_y\right|^2+\left|v'_x\right|^2+\left|v'_y\right|^2\right\}\{s^2+t^2\}}, \tag{8}$$

where all partial derivatives are calculated at the point $(a+\xi s, b+\xi t)$.

Let $f(z)=u(x,y)+jv(x,y)$ is h-holomorphic function, then $u'_x=v'_y,\ \ u'_y=v'_x$. Consequently

$$f'(z)=u'_x+jv'_x=u'_x+ju'_y=v'_y+jv'_x=v'_y+ju'_y,$$

$$\left|f'(z)\right|=\sqrt{|u'_x|^2+|v'_x|^2}\le\left|u'_x\right|+\left|u'_y\right|=\left\|f'(z)\right\|,$$

$$\|f(z+h)-f(z)\|=\|\triangle u+j\triangle v\|=|\triangle u|+|\triangle v|\le\sqrt{2}\cdot\sqrt{|\triangle u|^2+|\triangle v|^2},$$

where $|h|=|s+jt|=\sqrt{s^2+t^2}\le|s|+|t|=\|h\|$.

Theorem 6 (on finite increments for an *h*-holomorphic function). *Let the function f be h-holomorphic in the domain $D\subset\mathbb{C}_h$. Then*

$$\|f(z+h)-f(z)\|\le 2\max_{\zeta\in[z,\ z+h]}\left\|f'(\zeta)\right\|\ \|h\|$$

P r o o f. Due to the inequality (7), we have

$$\|f(z+h)-f(z)\|\le\sqrt{2}\cdot\sqrt{|\triangle u|^2+|\triangle v|^2}\le\max_{\xi\in[0,1]}\sqrt{2\left\{\left|u'_x\right|^2+\left|u'_y\right|^2\right\}\{s^2+t^2\}}\le$$
$$\le 2\max_{\zeta\in[z,\ z+h]}\left|f'(\zeta)\right|\ |h|\le 2\max_{\zeta\in[z,\ z+h]}\left\|f'(\zeta)\right\|\ \|h\|.$$

The theorem is proved.

h-analyticity of h-holomorphic functions

Definition 4. A function f is called h-analytic at a point $z_0\in D$ if there exists a neighborhood of this point, where f expands into a convergent power series

$$f(z)=\sum_{k=0}^{\infty}c_k(z-z_0)^k. \tag{9}$$

The definition implies that the function f is infinitely h-differentiable in some neighborhood of the point z_0 and the series (9) is the Taylor series of the function f, i. e. $c_k=\frac{f^k(z_0)}{k!}$. The convergence domain of the series (9) is an open h-circle

$$G=\{\|z-z_0\|<r\},\qquad r=\frac{1}{\varlimsup\limits_{k\to\infty}\sqrt[k]{c_k}}.$$

Theorem 7. *Let the function $f:D\to\mathbb{C}_h$ be infinitely many times h-differentiable in the domain $D\subset\mathbb{C}_h$,*

$$\left\|f^{(n)}(z)\right\|\le Me^{AR^m}\qquad\forall n\in\mathrm{N},\ \ \forall z\in\{\|z-z_0\|\le R\}\subset D, \tag{10}$$

$M, A,\ m$ are some positive constants. Then f expands into a Taylor series

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,\qquad z_0\in D,$$

uniformly convergent in the circle $\|z-z_0\|\le R$.

P r o o f. We represent $f(z)$ as

$$f(z) = T_n(z, z_0) + r_n(z),$$

where $T_n(z, z_0) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$, $r_n(z)$ is remainder term. Let's compose an auxiliary function

$$F(t) = f(z) - T_n(z, t) = f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(z - t)^k.$$

For it we have $F(z) = 0,\ F(z_0) = r_n(z)$. Differentiate $F(t)$ by variable t

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(z - t)^n.$$

Due to the Theorem 6 and condition (10), we obtain

$$\|r_n(z)\| = \|F(z_0) - F(z)\| \le 2 \max_{\zeta \in [z,\, z+h]} \left\|F'(\zeta)\right\| \cdot \|z_0 - z\| \le 2 \max_{\zeta \in [z,\, z+h]} \left\| \frac{f^{(n+1)}(t)}{n!}(z - t)^n \right\| \cdot \|z_0 - z\| \le$$

$$\le 2 \sup_{\zeta \in [z,\, z+h]} \frac{1}{n!} \left\|f^{(n+1)}(\zeta)\right\| \cdot \|(z - t)^n\| \cdot \|z_0 - z\| \le \frac{2}{n!} M e^{AR^m} R^{n+1} \underset{n \to \infty}{\longrightarrow} 0$$

provided $\|z - t\| \le R$ and $\|z - z_0\| \le R$. From here we deduce

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,$$

where the series converges uniformly in the circle $\|z - z_0\| \le R$. The theorem is proved.

Corollary 1. The remainder term of the Taylor formula in the Peano form is

$$r_n(z) = o\left(\|z - z_0\|^n\right).$$

Definition 5. Function f is h-analytic in the domain $D \in \mathbb{C}_h$ if it is h-analytic at all points of this domain.

Let f be h-analytic at the point z_0 therefore in a neighborhood of the point z_0 we have

$$f(z) = c_k (z - z_0)^k + c_{k+1} (z - z_0)^{k+1} + \ldots, \tag{11}$$

where $c_k \neq 0$, $k \ge 0$.

Definition 6. Point z_0 is called a zero of order k function f if in (11) $k \ge 1$.

From (11) implies the representation

$$f(z) = (z - z_0)^k \cdot \varphi(z),$$

where $\varphi(z) = c_k + c_{k+1}(z - z_0) + \ldots$, $\varphi(z)$ is h-analytic in a neighborhood of the point z_0, $\varphi(z_0) = c_k \neq 0$. Due to the continuity of the function $\varphi(z)$, there exists a neighborhood $U(z_0)$: $\varphi(z) \neq 0\ \ \forall\, z \in U(z_0)$. This implies the following theorem.

Theorem 8. *If f is expandable in a neighborhood of the point z_0 in a series (11), where $k \ge 1$, and c_k is not a zero divisor, then there is a neighborhood of the point z_0 in which f has no other zeros, besides z_0.*

Theorem 9 (uniqueness theorem for *h*-analytic functions). *Let f_1 and f_2 are h-analytic in the domain $D \subset \mathbb{C}_h$, $f_1(z) \equiv f_2(z)\ \ \forall z \in E \subset D$, where E has a limit point in D and does not contain zero divisors. Then $f_1(z) \equiv f_2(z)$ everywhere in D.*

P r o o f. We denote

$$f(z) = f_1(z) - f_2(z).$$

Let $\zeta \in D$ be the limit point of the set E. Let's choose the sequence $\zeta_k \in E$: $\lim\limits_{k \to \infty} \zeta_k = \zeta$. Due to continuity,

$$f(\zeta) = \lim_{k \to \infty} f(\zeta_k) = 0.$$

Theorem 8 implies that $f(z) \equiv 0$ in some neighborhood of the point ζ. Let $M \subset D$ be the set of zeros of the function f, $\overset{\circ}{M}$ its interior. From the above it follows that $\overset{\circ}{M} \neq \emptyset$. If $\overset{\circ}{M} = D$, then the theorem is proved. If $\overset{\circ}{M} \underset{\neq}{\subset} D$, then there is a boundary point d of the set $\overset{\circ}{M}$, which is an interior point of the set D. Then there exists a sequence $d_n \in \overset{\circ}{M}$: $\lim\limits_{n\to\infty} d_n = d$. Due to continuity,

$$f(d) = \lim_{n\to\infty} d_n = 0.$$

On the other hand, $f(z)$ is not identically equal to zero in any neighborhood of the point d, since d is not an interior point, but a boundary point of the set $\overset{\circ}{M}$. Theorem 8 implies that in some neighborhood of the point d there are no other zeros of the function f, except d. This contradicts the fact that d is a boundary point of the set $\overset{\circ}{M}$. From this we conclude that $\overset{\circ}{M} = D$. The theorem is proved.

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h-комплекс айнымалы функциялардың h-голоморфтылығы және h-аналитикалығы

Аннотация: Соңғы кездерде геометрия мен механикадағы қолданыстарына байланысты h-комплекс сандар жиынында анықталған функциялардың қасиеттерін зерттеу жұмыстарына қызығушылық арта бастады. Ұсынылған мақалада h -комплекс айнымалы функцияның h - дифференциалдануы және h -голоморфтiлiгi үшiн қажеттi және жеткiлiктi шарттар келтiрiлген; ақырлы өсiмшелер туралы теорема дәлелдендi; функцияның h-аналитикалық болуының жеткiлiктi шарттары табылды; h-аналитикалық функциялар үшiн жалғыздық теоремасы дәлелдендi.

Түйiн сөздер: h-дифференциалдану, h-голоморфтылық, h-аналитикалық, h-комплекс сандар сақинасы, нөлдiң бөлгiштерi, функцияның нөлдерi.

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Об h-голоморфности и h-аналитичности функций h-комплексного переменного

Аннотация: В последнее время в связи с имеющимися приложениями в геометрии и механике возрос интерес к исследованиям свойств функций, заданных на множестве h-комплексных чисел. В представленной статье приводятся необходимые и достаточные условия h-дифференцируемости и h-голоморфности функций h-комплексного переменного; доказана теорема о конечных приращениях; найдены достаточные условия h-аналитичности; доказана теорема единственности для h-аналитических функций.

Ключевые слова: h-дифференцируемость, h-голоморфность, h-аналитичность, Кольцо h-комплексных чисел, Делители нуля, Нули функции.

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