Л.Н. Гумилев атындағы Еуразия ұлттық университетінің хабаршысы. Математика. Компьютерлік ғылымдар. Механика сериясы, 2020, том 131, №2, 8-27 беттер
http://bulmathmc.enu.kz, E-mail: vest_math@enu.kz

МРНТИ: 30.19.33, 30.19.21

L.A. Alexeyeva, M.M. Akhmetzhanova

Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan
(E-mail: alexeeva@math.kz, mariella80@mail.ru)

Method of generalized functions in boundary value problems of thermoelastic rod dynamics

Abstract: The method of generalized functions (GFM) has been developed to solve transient and vibrational boundary value problems of thermoelastic rod dynamics using a model of coupled thermoelasticity. Thermoelastic shock waves arising in such structures under the influence of shock loads and heat flows are considered. Conditions on their fronts were obtained. The singularity of the assigned boundary tasks taking into account shock waves has been proved. On the basis of GFM, a system of algebraic resolving equations is built for a wide class of boundary problems to determine their analytical solutions. Dynamics of the rod under the action of forces and heat sources of various types, including those described by singular generalized functions, which allow modeling the effect of pulsed concentrated sources, are studied. Computer implementation of solutions of one edge problem at stationary oscillations was carried out, results of numerical experiments of calculation of rod thermodynamics at low and high frequencies are presented. These solutions and algorithms can be used for engineering calculations of rod structures to evaluate their strength properties.

Keywords: thermoplasticity, rod, boundary value problems,stress-strain state, general functions method.

DOI: https://doi.org/10.32523/2616-7182/2020-131-2-8-27

Introduction. Rod structures are widely used in mechanical engineering as connecting and transmission links for structural elements of a wide variety of machines and mechanisms. During operation, they are subjected to variable mechanical and thermal stresses that create a complex stress-strain state in structural elements, depending on their temperature, and affecting their strength and reliability. Therefore, the determination of a thermal stress state of rod structures taking into account their mechanical properties (in particular, elasticity) is one of the urgent scientific and technical problems.

When studying thermodynamic processes in structures, equations of uncoupled thermoelasticity are usually used. In this model at first the temperature problem is solved for determining the temperature field without taking into account the deformation of medium. This reduces a problem to constructing a solution of boundary value problem (BVP) for the heat parabolic equation. After determining a temperature field, BVP of dynamics of thermo-elastic medium is solved, in which a gradient of known temperature field is introduced as a mass force in motion equations of elastic medium. This model describes thermodynamic processes well at low strain rates and is completely unsuitable for describing high-speed dynamic processes.

Here, a problem of determining a thermostressed state of a thermoelastic rod is considered, using a model of coupled thermoelasticity. In this case, a heat equation contains a divergence of a velocity of material points of a medium, and a temperature gradient is included in equations of elasticity. This connects equations into one system of differential equations of mixed type without separating a temperature field and elastic deformations.

Note that nonstationary BVPs of coupled thermoelastodynamics by plane deformation and in 3D-space were considered by authors [1-7] and others. They elaborated analytical Boundary Integral equations Method and numerical Boundary Elements Method for construction BVP solutions in a space of Laplace or Fourier transformation over time. In [3] BIEM is based on potentials theory. In [7] BIEM was elaborated by use General Functions Method which is essentially convenient for solving hyperbolic and mixed problems of mathematical physics. The base ideas of this method are presented in paper [8].

Here we elaborate this method for solving non-stationary BVPs and stationary vibrations problems of dynamics of a thermoelastic rod under the action of power and heat sources of various types, including those described by singular generalized functions. The latter allows to simulate the impact of pulsed concentrated sources of various types. Thermal shock waves that arise in such structures under action of shock loads and heat fluxes are considered, and conditions at their fronts are obtained. Uniqueness of posed boundary value problem is proved, subject to shock waves. Based on GFM, algebraic resulting equations system for wide class of boundary value problems have been constructed for determination of analytical solutions of BVPs. As example the computer implementation of solutions of one BVP was carried out by stationary oscillations at low and high frequencies The results of some computer experiments have been presented.

1. Statement of non-stationary boundary value problems of connected thermoelasticity. A thermoelastic rod of length $2L$ are considered, which is characterized by a density ρ, rigidity EJ, and thermoelastic constants γ, η and κ [1,2].The movement of the cross sections of the rod and the temperature field of the rod is described by a system of hyperbolic-parabolic equations of the form:

$$\rho c^2 u,_{xx} - \rho u,_{tt} - \gamma \theta,_x + \rho F_1 = 0,$$

$$\theta,_{xx} - \kappa^{-1} \theta,_t - \eta u,_{xt} + F_2 = 0. \tag{1.1}$$

Here $u(x,t)$ are the components of the longitudinal displacements, $\theta(x,t)$ are the relative temperature $(\theta = T(x,t) - T(x,0))$, T are absolute temperature, F_1 are a longitudinal component of acting forces; a velocity of thermoelastic waves propagation $c = \sqrt{\frac{EJ}{\rho}}$. An action of heat sources describes by the function $F_2 = (\lambda_0 \kappa)^{-1} W(x,t)$, where W are amount of released (or absorbed) heat per unit volume per unit time, λ_0 is a thermal conductivity coefficient.

We suppose that functions $F_1(x,t), F_2(x,t)$ belong to a space of generalized functions (distributions) of slow growth S [9], that allows us to simulate thermodynamic processes in rods under action of various types of concentrated heat sources. Hereinafter, we use the notation for partial derivatives: $u_i,_j = \partial u_i / \partial x_j = \partial_j u_i$. Thermoelastic stress in the rod is determined by the Duhamel-Neumann relation [1,2]:

$$\sigma = \rho c^2 u,_x - \gamma \theta \tag{1.2}$$

We consider a number of direct boundary value problems of thermoelasticity whose solutions satisfy the following initial and boundary conditions. *Initial conditions* (Cauchy conditions): at $t = 0$ the displacement, velocity and temperature are known:

$$\begin{gathered} u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \quad |x| \le L; \\ \partial_t u(x,0) = \dot{u}_0(x), \quad |x| < L \end{gathered} \tag{1.3}$$

Boundary conditions at the rod ends $(x = x_1 = -L,\ x = x_2 = L)$ depend on BVP type. Here at first we consider four classic BVPS.

BVP I. A displacement and temperature at rod ends are known:

$$u(x_j,t) = w_j(t), \quad \theta(x_j,t) = \theta_j(t); \quad j = 1,2 \tag{1.4}$$

BVP II. Stresses and heat fluxes at rod ends are known:

$$\sigma(x_j,t) = p_j(t), \quad \theta_{,x}(x_j,t) = q_j(t); \quad j = 1,2 \tag{1.5}$$

BVP III. A displacement and heat fluxes at rod ends are known:

$$u(x_j,t) = w_j(t), \quad \theta_{,x}(x_j,t) = q_j(t); \quad j = 1,2 \tag{1.6}$$

BVP IV. Stresses and temperature at rod ends are known:

$$\sigma(x_j,t) = p_j(t), \quad \theta(x_j,t) = \theta_j(t); \quad j = 1,2 \tag{1.7}$$

It is assumed that the boundary functions satisfy the following smoothness conditions:

$$u_j(t) \in C(0,\infty), \theta_j(t) \in C(0,\infty), \ q_j(t) \in L_1(0,\infty), \ p_j(t) \in L_1(0,\infty) \tag{1.8}$$

and are regular functions from $S'\left(R^1\right)$.

Remark. By $\eta = 0$ it is the model of uncoupled thermoelasticity, by $\gamma = 0$ the first equation (1.1) is the motion equation of elastic rods.

2. Shock thermoelastic waves as generalized solutions of motion equations. The system of equations (1.1) has mixed hyperbolic-parabolic type. Due to a hyperbolic personality, it's possible an occurrence of thermoelastic shock waves by cause shock effects at ends of a rod. To derive shock waves, we consider Eqs (1.1) and their solutions in a space of distributions S'.

Let $u(x,t), \theta(x,t)$ are classic solution of Eqs(1.1). We consider them as regular distributions, which are differentiable between fronts of shock waves, where there derivatives are discontinues . According to the rules of differentiation of such generalized functions [9], Eqs (1.1) for thermoelastic shock waves take the form in S':

$$\begin{aligned} &\rho c^2 u_{,xx} - \rho u_{,tt} - \gamma\theta_{,x} + F_1 + \left(\left[\rho c^2 u_{,x} - \gamma\theta\right]\nu_x - \rho\left[u_{,t}\right]\nu_t\right)\delta_F(x,t) + \\ &+\partial_x\left[\rho c^2 u\right]\delta_F(x,t) - \partial_t\left[\rho u\right]\delta_F(x,t) = 0, \\ &\theta_{,xx} - \kappa^{-1}\theta_{,t} - \eta u_{,xt} + F_2 + \partial_x\left[\theta\right]\nu_x\delta_F + \left[\theta_{,x}\right]\nu_x\delta_F - \\ &-\left[\kappa^{-1}\theta + \eta u_{,x}\right]\nu_t\delta_F - \partial_t\left[\eta u\right]\nu_x\delta_F = 0. \end{aligned} \tag{2.1}$$

Here, the square brackets denote the jump of functions indicated in them at the fronts of shock waves, $\delta_F(x,t)$ is singular generalized function – a simple layer on characteristic surface F in the set $D^- = \{(x,\tau) : |x| < L, \ \tau < t\}$, on which derivatives have jumps. As follow from (1.1), the next determinant vanishes on F:

$$\begin{vmatrix} \rho\left(c^2\nu_x^2 - \nu_t^2\right) & 0 \\ \nu_x^2 & -\eta\nu_t\nu_x \end{vmatrix} = -\eta\rho\nu_t\nu_x\left(c^2\nu_x^2 - \nu_t^2\right) = 0 \tag{2.2}$$

where $\nu = (\nu_x, \nu_t)$ are the normal to F in D^-. It follows from (1.7) that the lines $x = const$ and $t = const$ are characteristic surfaces for equations (1.1), and for shock waves (F_t):

$$\nu_t = -c\left|\nu_x\right| \tag{2.3}$$

Here the wave front F_t has a simple form:

$$F_t = \left\{(x,t) : x \pm ct) = x^0\right\}$$

It is the point of derivatives discontinuity which moves at a speed c from the point x^0, where it is formed, in one direction along the rod or another.

As in a domain of differentiability, shock waves are solutions of Eqs. (1.1), from (2.1), taking into account (1.2), to be generalized solution of (1.1) it's necessary to perform next equalities:

$$\begin{aligned} &\left(\left[\rho c^2 u_{,x} - \gamma\theta\right]\nu_x - \rho\left[u_{,t}\right]\nu_t\right)\delta_F + \partial_x\left\{\left[\rho c^2 u\right]\delta_F\right\} - \partial_t\left\{\left[\rho u\right]\delta_F\right\} = 0 \\ &\partial_x\left\{\left[\theta\right]\nu_x\delta_F\right\} + \left(\left[\theta_{,x}\right]\nu_x - \left[\kappa^{-1}\theta + \eta u_{,x}\right]\right)\nu_t\delta_F = 0 \end{aligned} \tag{2.4}$$

From (2.4), taking into account (2.3), it follows that at the fronts of shock waves the following conditions for jumps must be satisfied:

$$[u]_{F_t} = 0, \quad [\sigma]_{F_t} = -\rho c\,[\dot{u}]_{F_t} \tag{2.5}$$

$$[\theta]_{F_t} = 0, \quad [\theta,_x]_{F_t} = \eta\,[\dot{u}]_{F_t} \tag{2.6}$$

The first condition (2.5) is continuity of displacements which is necessary to conserve continuity of a medium. The second condition describes a stress jump (shock), which leads to a jump in velocity at the wave front. From the first and second conditions (2.6) it follows that the temperature is continuous at the wave fronts but a heat flux has a jump proportional to a jump in displacements velocity at wave front.

From these relations follow that a jump in a heat flux in the rod also forms a thermoelastic shock wave, since it causes a jump in velocities at the front, which leads to a jump in stresses on it. Such thermo-shock waves are always formed at the ends of rod if, until a fixed point in time, it was in a static state, and then non-zero stresses or heat fluxes , applied to it at the ends, create thermoelastic shock waves.

3. Uniqueness of BVP solution subject to shock waves. We show uniqueness of the solution of the initial-boundary value problem in presence of shock waves. It is assumed that at each fixed point in time, the domain of solution determination with respect to x are divided into a finite number of intervals between the fronts of shock waves F_t^k at which the solution is continuous and differentiable according to (2.1). Denote an energy density of a rod

$$E(x,t) = 0,5\left\{\rho\,(u,_t)^2 + c^2\,(u,_x)^2 + \gamma\,(\eta\kappa)^{-1}\,\theta^2\right\}$$

and power of internal forces:

$$M(x,t) = u,_t\left(c^2 u,_x - \gamma\theta\right) + \eta\gamma^{-1}\theta\theta,_x .$$

Further we assume $\|\nu\| = 1$. From (2.2) it follows: $\nu = (\nu_x, \nu_t) = (1, -c)/\sqrt{1+c^2}$. The following theorem is true.

Theorem 1 *(law of conservation of energy)*

$$\int_{-L}^{L} \left(E(x,t) - E(x,0)\right) dx = \int_0^t dt \int_{-L}^{L} \left(u,_t F_1 + \eta\gamma^{-1}\theta F_2\right) dx +$$

$$+ \int_0^t \left(M(L,t) - M(-L,t)\right) dt - \eta\gamma^{-1} \int_0^t dt \int_{-L}^{L} (\theta,_x)^2 dx$$

.

Proof. We fix an arbitrary time $t > 0$. Multiplying the first equation (1.1) in the field of differentiability by $u,_t$, and the second equation by $\alpha\theta$, after a series of equivalent transformations, we obtain the equalities:

$$\rho c^2 u,_t\, u,_{xx} - u,_t\, u,_{tt} - \gamma u,_t\, \theta,_x + \rho F_1 u,_t = 0 \Rightarrow$$

$$\partial_x\left(u,_t\left(\rho c^2 u,_x - \gamma\theta\right)\right) - 0,5\partial_t\left\{(u,_t)^2 + \rho c^2\,(u,_x)^2\right\} + \gamma u,_{tx}\,\theta + u,_t\,\rho F_1 = 0;$$

$$\theta\theta,_{xx} - \kappa^{-1}\theta\theta,_t - \eta\theta u,_{xt} + \theta F_2 = 0 \Rightarrow$$

$$-0,5\kappa^{-1}\partial_t\theta^2 + \partial_x\left(\theta\theta,_x\right) - \eta\theta u,_{xt} - (\theta,_x)^2 + \theta F_2 = 0$$

Folding them, we have

$$\partial_x\left(u,_t\left(\rho c^2 u,_x - \gamma\theta\right) + \alpha\theta\theta,_x\right) -$$

$$-0,5\partial_t\left\{\rho\,(u,_t)^2 + c^2\,(u,_x)^2 + \alpha\kappa^{-1}\theta^2\right\} - \alpha\,(\theta,_x)^2 +$$

$$+\theta u,_{xt}\,(\gamma - \alpha\eta) + u,_t\,\rho F_1 + \alpha\theta F_2 = 0.$$

where $\alpha = \eta/\gamma$. As a result, we obtain the equality:

$$\partial_t E(x,t) - \partial_x M(x,t) + \eta\gamma^{-1}\,(\theta,_x)^2 = u,_t\, F_1 + \eta\gamma^{-1}\theta F_2 \tag{3.1}$$

Lets integrate (3.1) over D^- with allowance for the division of integration region by the fronts of shock waves $F_k(x,t)$ into subdomains where the solution is differentiable. As a result, using the Ostrogradsky-Gauss theorem, we obtain the following integral equality:

$$\begin{gathered}\int\limits_{-L}^{L} (E(x,t)-E(x,0))\,dx+\alpha\int\limits_0^t dt\int\limits_{-L}^{L}(\theta,_x)^2dx=\\ =\int\limits_0^t (M(L,t)-M(-L,t))\,dt+\int\limits_0^t dt\int\limits_{-L}^{L}(u,_t\rho F_1+\alpha\theta F_2)\,dx+\\ +\left\{\int\limits_{F_k}\sum_k\left[\nu_x M(x,t)-\nu_t E(x,t)\right]_{F_k}dS(F_k)\right\}.\end{gathered} \tag{3.2}$$

We show that, due to conditions at the fronts of shock waves (2.5)-(2.6), the jumps on the right-hand side of this equality are equal to zero. To do this, we make a series of transformations:

$$\begin{gathered}[M(x,t)]_{F_k}=[u,_t\sigma]_{F_k}+\alpha\,[\theta\theta,_x]_{F_k}=u^-,_t[\sigma]_{F_k}+\sigma^+[u,_t]_{F_k}+\alpha\theta\,[\theta,_x]_{F_k}=\\ =\left(\sigma^+-\rho c u^-,_t+\gamma\theta\right)[u,_t]_{F_k}=\rho c\left(cu,_x^+-u^-,_t\right)[u,_t]_{F_k}\end{gathered}$$

(here the signs in the upper index indicate the values of the corresponding functions on the right or left side of the wave front). Consequently,

$$\begin{gathered}\sqrt{1+c^2}\,[\nu_x M(x,t)-\nu_t E(x,t)]_{F_k}=[M(x,t)+cE(x,t)]_{F_k}=\\ =\rho c\left(cu,_x^+-u^-,_t\right)[u,_t]_{F_k}-\rho c\,[u,_t]_{F_k}\left(cu^-,_x-u,_t^+\right)=\rho c\,[cu,_x+u,_t]\,[u,_t]_{F_k}=0\end{gathered}$$

since, in virtue (2.5),

$$\begin{gathered}[cu,_x+u,_t]=\frac{1}{\rho c}\left(\left[\rho c^2u,_x-\theta\right]_{F_t}+\rho c\,[\dot u]_{F_t}\right)=\\ =\frac{1}{\rho c}\,[\sigma+\rho c\dot u]_{F_t}=0\end{gathered}$$

Therefore, from (3.2) we obtain the formula of the theorem.

Theorem 2. *The solutions of BVPs I-IV are unique.*

Proof. We carry out the opposite. Let there exist two solutions of the considered BVP from the stated ones. Then their difference, by virtue of linearity, will also be a solution of (1.1) for $F_j=0,\ j=1,2$, and satisfy zero initial and boundary conditions. We write the energy conservation law for such solution. According to Theorem 1:

$$\int\limits_{-L}^{L}E(x,t)dx+\eta\gamma^{-1}\sqrt{1+c^{-2})}\int\limits_0^t dt\int\limits_{-L}^{L}(\theta,_x)^2dx=\int\limits_0^t(M(L,t)-M(-L,t))\,dt$$

But

$$\int\limits_0^t M(\pm L,t)\,dt=\int\limits_0^t\left(u,_t(\pm L,t)\sigma(\pm L,t)+\eta\gamma^{-1}\theta(\pm L,t)\theta,_x(\pm L,t)\right)dt=0$$

since by one of the factors in each integrand is equal to zero, due to the zero boundary conditions of any BVP. Therefore

$$\int\limits_{-L}^{L}E(x,t)dx+\eta\gamma^{-1}\int\limits_0^t dt\int\limits_{-L}^{L}(\theta,_x)^2dx=0$$

Due to the zero initial conditions and the positive definiteness of the integrands, we obtain $u\equiv 0,\ \theta\equiv 0$. Then decisions are coincided. The theorem is proved.

4. Generalized solution of BVP. To determine the solution, we pose a boundary value problem in the space of two-dimensional generalized vector functions

$$S_2'(R^2)=\{\hat f=(\hat f_1(x,t),\hat f_2(x,t)),\quad (x,t)\in R^2,\quad \hat f_j\in S'(R^2),j=1,2\}$$

Their components are generalized functions which belong to $S'(R^2)$ [3]). To do this, we introduce a generalized regular vector function (mark them with a hat):

$$(\hat{u}_1, \hat{u}_2) = \{\hat{u}, \hat{\theta}\} = \{u(x,t)H\ (x)H(t)\,, \theta(x,t)H\ (x)H(t)\}$$

Here $(u_1, u_2) = (u(x,t), \theta(x,t))$ are the solution of BVP, $H(x)$ are the Heaviside function.
In $S_2'(R^2)$ vector-function $(\hat{u}_1, \hat{u}_2)$ satisfies to the next system:

$$\begin{array}{l} c_1^2\hat{u}_{,xx} - \hat{u}_{tt} - \breve{\gamma}\hat{\theta}_{,x} + \hat{F}_1 = -\{\dot{u}_0(x)\delta(t) + u_0(x)\delta'(t)\}\, H(L-|x|) + \\ + c_1^2 H(t)\{(p_1(t) - \gamma\theta_1(t))\,\delta(x+L) - (p_2(t) - \gamma\theta_2(t))\,\delta(x-L)\} + \\ + c_1^2 H(t)\{u_1(t)\delta'(x+L) - u_2(t)\delta'(x-L)\}\,, \\ \hat{\theta}_{,xx} - \kappa^{-1}\hat{\theta}_{,t} - \eta\hat{u}_{,xt} + \hat{F}_2 = \\ = H(t)\,\delta(L+x)\,(q_1(t) - \eta\dot{u}_1(t)) - H(t)\,\delta(L-x)\,(q_2(t) - \eta\dot{u}_2(t)) + \\ + \left(\hat{\theta}_1(t)H(t)\,\delta'(L+x)\right) - \left(\hat{\theta}_2(t)H(t)\,\delta'(L-x)\right) - \kappa^{-1}\hat{\theta}_0(x)\delta(t)\,H(L-|x|) - \\ - \eta\delta(t)\,H(L-|x|)\,\partial_x\dot{u}_0(x) - \eta u_1(0)\delta(t)\,\delta(L+x) + \eta u_2(0)\delta(t)\,\delta(L-x) \end{array} \tag{4.1}$$

Here are $\delta(t)$ is singular delta - function, $\breve{\gamma} = \gamma/\rho$.
Using the property of the matrix of fundamental solutions $\hat{U}_j^k(x,t)$, the solution of Eqs(4.1) can be written as following tensor-functional convolution:

$$\begin{array}{l} u(x,t)\,H(t)\,H(L-|x|) = \hat{F}_1 * \hat{U}_1^1 + \hat{F}_2 * \hat{U}_1^2 + \\ + c^2 \sum\limits_{k=1}^{2} (-1)^{k+1}\left\{(p_k(t) - \gamma\theta_k(t)) \underset{t}{*} U_1^1(x+L,t) + u_k(t) \underset{t}{*} U_1^1{}_{,x}(x+L,t)\right\} + \\ + H(t) \sum\limits_{k=1}^{2} (-1)^{k+1}(q_k(t) - \eta\dot{u}_k(t)) \underset{t}{*} \hat{U}_1^2\left(x - (-1)^k L\right) + \theta_k(t)H(t) \underset{t}{*} U_1^2{}_{,x}(x+L) - \\ - \left\{\dot{u}_0(x) \underset{x}{*} \hat{U}_1^1(x,t) + u_0(x)) \underset{x}{*} \hat{U}_1^1{}_{,t}(x,t\right\} H(L-|x|) - \\ - \eta\, u_1(0)\, U_1^2(L+x,t) + \eta\, u_2(0)\, U_1^2(x-L,t) - \\ - \kappa^{-1}\theta_0(x)H(L-|x|) \underset{x}{*} U_1^2 - \eta H(L-|x|)\,\partial_x\dot{u}_0(x) \underset{x}{*} U_1^2 \end{array} \tag{4.2}$$

$$\begin{array}{l} \theta(x,t)\,H(t)\,H(L-|x|) = \hat{F}_1 * \hat{U}_2^1 + \hat{F}_2 * \hat{U}_2^2 + \\ + c^2 \sum\limits_{k=1}^{2} (-1)^{k+1}\left\{(p_k(t) - \gamma\,\theta_k(t)) \underset{t}{*} U_2^1(x+L,t) + u_k(t) \underset{t}{*} U_2^1{}_{,x}(x+L,t)\right\} + \\ + H(t) \sum\limits_{k=1}^{2} (-1)^{k+1}(q_k(t) - \eta\,\dot{u}_k(t)) \underset{t}{*} \hat{U}_2^2\left(x - (-1)^k L\right) + \theta_k(t)H(t) \underset{t}{*} U_2^2{}_{,x}(x+L) - \\ - \left\{\dot{u}_0(x) \underset{x}{*} \hat{U}_2^1(x,t) + u_0(x)) \underset{x}{*} \hat{U}_2^1{}_{,t}(x,t)\right\} H(L-|x|) - \\ - \eta u_1(0)U_2^2(L+x,t) + \eta u_2(0)U_2^2(x-L,t) - \\ - \kappa^{-1}\theta_0(x)H(L-|x|) \underset{x}{*} U_2^2 - \eta H(L-|x|)\,\partial_x\dot{u}_0(x) \underset{x}{*} U_2^2 \end{array} \tag{4.3}$$

The matrix of fundamental solutions $U_i^j(x,t)\ (i,j=1,2)$ is solution (1.1) for singular

$$F = (F_1, F_2) = \delta_i^j\delta(x)\delta(t)$$

δ_i^j is Kronecker symbol. The integral record of convolutions (4.2), (4.3) has the next form:

$$u(x,t)\,H(|x|-L)H(t) = \hat{F}_1 * \hat{U}_1^1 + \hat{F}_2 * \hat{U}_1^2 +$$

$$+c^2 H(t)\sum_{k=1}^{2}(-1)^{k+1}\int\limits_0^t \{(p_k(\tau) - \widetilde{\gamma}\theta_k(\tau))\,U_1^1(x-(-1)^k L, t-\tau) +$$

$$\left. + u_k(\tau)U_1^1{}_{,x}(x-(-1)^k L, t-\tau)\right\} d\tau +$$

$$+H(t)\sum_{k=1}^{2}(-1)^{k+1}\int_{0}^{t}\Big\{\left(q_k(\tau)-\eta\dot{u}_k(\tau)\right)U_1^2(x-(-1)^kL,t-\tau)+ \tag{4.4}$$

$$+\theta_k(\tau)U_1^2{,_x}\left(x-(-1)^kL,t-\tau\right)\Big\}\,d\tau-$$

$$-H\left(L-|x|\right)\int_{-L}^{L}\left(\dot{u}_0(y)U_1^1(x-y,t)+u_0(y)\right)U_{1,t}^1(x-y,t)dy-$$

$$-\eta u_1(0)U_1^2\left(L+x,t\right)+\eta u_2(0)U_1^2\left(x-L,t\right)-$$

$$-H\left(L-|x|\right)\int_{-L}^{L}\left\{\kappa^{-1}U_1^2(x-y,t)\theta_0(y)-\eta U_1^2(x-y,t)\partial_y\dot{u}_0(y)\right\}dy.$$

$$\theta\left(x,t\right)H\left(t\right)H\left(L-|x|\right)=\hat{F}_1*\hat{U}_2^1+\hat{F}_2*\hat{U}_2^2+$$

$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\int_{0}^{t}\left\{\left(p_k(\tau)-\widetilde{\gamma}\theta_k(\tau)\right)U_2^1(x+L,t-\tau)+\right.$$

$$\left.+u_k(t)U_2^1{,_x}\left(x+L,t-\tau\right)\right\}d\tau+$$

$$+H(t)\int_{0}^{t}\left\{\sum_{k=1}^{2}(-1)^{k+1}\left(q_k(\tau)-\eta\dot{u}_k(\tau)\right)U_2^2\Big(x-(-1)^kL,t-\tau\Big)+\right.$$

$$\left.+\theta_k(\tau)U_2^2{,_x}\left(x+L,t-\tau\right)\right\}d\tau- \tag{4.5}$$

$$-H\left(L-|x|\right)\int_{-L}^{L}\left\{\dot{u}_0(y)U_2^1(x-y,t)+u_0(y))U_2^1{,_t}\left(x-y,t\right)+\right.$$

$$\left.+\eta u_1(0)U_2^2\left(L+x-y,t\right)+\eta u_2(0)U_2^2\left(x-y-L,t\right)\right\}dy-$$

$$-H\left(L-|x|\right)\int_{-L}^{L}\left\{\kappa^{-1}\theta_0(y)U_2^2(x-y,t)+\eta U_2^2(x-y,t)\partial_y\dot{u}_0(y)\right\}dy.$$

For regular functions

$$\hat{F}_j*\hat{U}_i^j=H(t)H(|x|-L)\int_{0}^{t}\int_{-L}^{L}F_j(y,\tau)U_i^j(x-y,t-\tau)dyd\tau$$

For singular $\hat{F}_j$, wich are applied in physical applications [10], the definition of convolution should be used [9].

If a rod was at rest and the temperature was constant until the initial time, then the initial conditions are zero and the formulas are simplified.

$$u\left(x,t\right)H(|x|-L)H\left(t\right)=\hat{F}_1*\hat{U}_1^1+\hat{F}_2*\hat{U}_1^2+$$

$$+c^2H(t)\sum_{k=1}^{2}(-1)^{k+1}\int_{0}^{t}\Big\{\left(p_k(\tau)-\widetilde{\gamma}\theta_k(\tau)\right)U_1^1(x-(-1)^kL,t-\tau)+$$

$$+u_k(\tau)U_1^1{,_x}\left(x-(-1)^kL,t-\tau\right)\Big\}\,d\tau+ \tag{4.6}$$

$$+H\left(t\right)\sum_{k=1}^{2}(-1)^{k+1}\int_0^t\left\{\left(q_k(\tau)-\eta\dot{u}_k(\tau)\right)U_1^2\left(x-(-1)^kL,t-\tau,\right)+\right.$$

$$\left.+\theta_k(\tau)U_1^2{,_x}\left(x-(-1)^kL,t-\tau\right)\right\}d\tau$$

$$\theta\left(x,t\right)H\left(t\right)H\left(L-|x|\right)=\hat{F}_1*\hat{U}_2^1+\hat{F}_2*\hat{U}_2^2+$$

$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\int_0^t\left\{\left(p_k(\tau)-\widetilde{\gamma}\theta_k(\tau)\right)U_2^1(x+L,t-\tau)+\right.$$

$$\left.+u_k(t)U_2^1{,_x}\left(x+L,t-\tau\right)\right\}d\tau+ \tag{4.7}$$

$$+H\left(t\right)\int_0^t\left\{\sum_{k=1}^{2}(-1)^{k+1}\left(q_k(\tau)-\eta\dot{u}_k(\tau)\right)U_2^2\left(x-(-1)^kL,t-\tau\right)+\right.$$

$$\left.+\theta_k(\tau)U_2^2{,_x}\left(x+L,t-\tau\right)\right\}d\tau$$

Formulas (4.6) and (4.7) determine the displacement and temperature inside the rod from the known displacements, stresses, temperature, and heat fluxes at its ends.

5. The Green matrix and its Fourier transform over time. To construct matrix of fundamental solutions of equations of coupled thermo elastodynamics analytically it's possible only in Fourier or Laplace transform spaces over time. Fourier transformant over time of Green matrix $U_k^j(x,t)$ we constructed in [11]. It is fundamental solution of Eqs (1.1) which satisfied to radiation conditions.

Its components have the form:

$$\tilde{U}_1^j(x,\omega)=\frac{\delta_1^j sgn(x)}{2(\lambda_1-\lambda_2)}\left\{i\omega\kappa^{-1}\left(\frac{sinx\sqrt{\lambda_2}}{\sqrt{\lambda_2}}-\right.\right.$$

$$\left.\left.-\frac{sinx\sqrt{\lambda_1}}{\sqrt{\lambda_1}}\right)+\left(\sqrt{\lambda_1}sinx\sqrt{\lambda_1}-\sqrt{\lambda_2}sinx\sqrt{\lambda_2}\right)\right\}- \tag{5.1}$$

$$-\frac{\gamma\delta_2^j\mathrm{sgn}(x)}{2(\lambda_1-\lambda_2)}\left(\cos x\sqrt{\lambda_1}-\cos x\sqrt{\lambda_2}\right),j=1,2$$

$$\tilde{U}_2^j(x,\omega)=\frac{sgn(x)}{2(\lambda_1-\lambda_2)}\left\{i\omega\eta\delta_1^j\left(\cos x\sqrt{\lambda_1}-\right.\right.$$

$$\left.-cosx\sqrt{\lambda_2}\right)-\omega^2\left(\frac{sinx\sqrt{\lambda_1}}{\sqrt{\lambda_1}}-\frac{sinx\sqrt{\lambda_2}}{\sqrt{\lambda_2}}\right)\delta_2^j+ \tag{5.2}$$

$$\left.+c^2\left(\sqrt{\lambda_1}\sin x\sqrt{\lambda_1}-\sqrt{\lambda_2}\sin x\sqrt{\lambda_2}\right)\delta_2^j\right\},j=1,2$$

Here

$$\lambda_{1,2}(\omega)=\frac{\omega}{2c^2}\left\{(\omega+i\gamma\eta)+ic^2k^{-1}\pm\sqrt{\left(\omega+i\left(\gamma\eta+c^2k^{-1}\right)\right)^2-4i\omega c^2k^{-1}}\right\} \tag{5.3}$$

the roots of the characteristic equation of system, quadratic with respect to ξ^2:

$$\Delta(\xi,\omega)=(\xi^2-ik^{-1}\omega)(c^2\xi^2-\omega^2)-i\gamma\eta\xi^2\omega=c^2\left(\xi^2-\lambda_1\right)\left(\xi^2-\lambda_2\right)$$

They depend on only three thermodynamic parameters of the medium:

$$c,\quad \alpha=\gamma\eta,\quad \beta=c^2k^{-1},$$

dimension $[\alpha]=[\beta]=[\omega]$. In these options

$$\lambda_{1,2}(\omega) = \frac{\omega}{2c^2}\left\{\omega + i(\alpha+\beta) \pm \sqrt{(\omega + i(\alpha-\beta))^2 - 4\alpha\beta}\right\} \tag{5.4}$$

Their frequency asymptotic behavior is as follows:
a) at $\omega \to \infty$:

$$\lambda_1 \sim \frac{\omega^2}{c^2}, \quad \lambda_2 \sim \frac{i\omega\,\beta}{c^2}, \tag{5.5}$$

b) at $\omega \to 0$:

$$\lambda_1 sim \frac{3i\omega\ (\alpha+\beta)}{2c^2}, \quad \lambda_2 \sim \frac{i\omega\,(\alpha+\beta)}{2c^2}. \tag{5.6}$$

Riemann surface of the matrix ω are univalent, since the values of the components $\tilde{U}_k^j$ are independent of the choice of the sign of the radicals $\sqrt{\lambda_j(\omega)}$.

The features $\tilde{U}_2^j(x,\omega)$ are clearly demonstrated in figure 1, where the calculations of this matrix are presented for the following conditional parameters: $\gamma = 0.1, c = 1, k = 1, \eta = 1$ the real (blue line) and imaginary part (green line) of each component are shown here.

Remark. Matrix $\tilde{U}_2^j(x,\omega)$ may be used also by solving BVPs of harmonic vibrations by action of periodic over time external forces and thermo-sources.

6. Laplace transforms over time of Green matrix. To solve non-stationary boundary value problems, we should use the Laplace transform of the fundamental matrix $\bar{U}_1^j(x,p)$, which is obtained using the connection between the Fourier transform and the Laplace transform in time ($p \leftrightarrow\ -i\omega,\ \omega \leftrightarrow\ ip$):

$$\tilde{U}_1^j(x,p) = \frac{\delta_1^j \operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\times$$
$$\times\left\{-p\kappa^{-1}\left(\frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}} - \frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}}\right) + \left(\sqrt{\lambda_1}\sin x\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin x\sqrt{\lambda_2}\right)\right\} -$$
$$-\frac{\gamma\delta_2^j \operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right), \quad j = 1,2$$

$$\tilde{U}_2^j(x,p) = -\frac{\operatorname{sgn}(x)}{2(\lambda_1-\lambda_2)}\times$$
$$\times\left\{p\eta\delta_1^j\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right) - p^2\left(\frac{\sin x\sqrt{\lambda_1}}{\sqrt{\lambda_1}} - \frac{\sin x\sqrt{\lambda_2}}{\sqrt{\lambda_2}}\right)\delta_2^j +\right.$$
$$\left.+c^2\left(\sqrt{\lambda_1}\sin x\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin x\sqrt{\lambda_2}\right)\delta_2^j\right\}, \quad j = 1,2$$

where

$$\lambda_{1,2}(p) = -\frac{p}{2c^2}\left\{p + \alpha + \beta \pm \sqrt{(p + (\alpha-\beta))^2 + 4\alpha\beta}\right\}$$

The components $\tilde{U}_k^j(x,p)$ are regular and continuous at the point $x = 0$:

$$\tilde{U}_k^j(\pm 0,\omega) = \tilde{U}_k^j(0,\omega) = 0, \quad k,j = 1,2, \tag{6.1}$$

But their derivatives

$$\partial_x \tilde{U}_1^j(x,\omega) =$$
$$\left[\frac{(\lambda_1 - i\omega\kappa^{-1})}{(\lambda_1-\lambda_2)}\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right) + \cos x\sqrt{\lambda_2}\right]\frac{\operatorname{sgn}(x)}{2}\delta_1^j -$$
$$+\frac{\gamma}{2(\lambda_1-\lambda_2)}\left(\sqrt{\lambda_1}\sin|x|\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin|x|\sqrt{\lambda_2}\right)\delta_2^j$$

$$\partial_x \tilde{U}_2^j(x,\omega) = -\delta_1^j \frac{i\omega\eta\left(\sqrt{\lambda_1}\sin|x|\sqrt{\lambda_1} - \sqrt{\lambda_2}\sin|x|\sqrt{\lambda_2}\right)}{2(\lambda_1-\lambda_2)} +$$
$$-\delta_2^j \operatorname{sgn}(x)\left\{\frac{\omega^2-\lambda_1 c^2}{2(\lambda_1-\lambda_2)}\left(\cos x\sqrt{\lambda_1} - \cos x\sqrt{\lambda_2}\right) - c^2\cos x\sqrt{\lambda_2}\right\}$$

at this point suffers a break of the first kind:

$$\partial_x \bar{U}_1^j(\pm 0,p) = \pm 0,5\delta_1^j, \quad \partial_x \bar{U}_2^j(\pm 0,p) = \pm 0,5c^2\delta_2^j, \quad j=1,2 \tag{6.2}$$

(the upper sign corresponds to the left limit at zero, the lower right).

Remark. By $\eta = 0$ matrix $\tilde{U}_k^j(x,p)$ is fundamental for equations of uncoupled thermoelastodynamics. In this case its original has be constructed in [12].

7. Laplace transform of boundary value problems solution. Here we consider the initial boundary value problem with zero initial conditions. By use the property of Laplace transform of convolution we get Laplace transformants of generalized solution from (4.2)-(4.3):

$$\begin{aligned}
&\bar{u}\,(x,p)\,H(|x|-L) = \bar{F}_1(x,p)*\bar{U}_1^1(x,p) + \bar{F}_2(x,p)*\bar{U}_1^2(x,p)+\\
&+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k - \breve{\gamma}\bar{\theta}_k\right)\bar{U}_1^1(x-(-1)^kL,p) + \bar{u}_k(\tau)\bar{U}_1^1{}_{,x}\,(x-(-1)^kL,p)\right\}+\\
&+\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{q}_k - \eta p\bar{u}_k\right)\bar{U}_1^2\left(x-(-1)^kL,p\right) + \bar{\theta}_k\bar{U}_1^2{}_{,x}\left(x-(-1)^kL,p\right)\right\}
\end{aligned} \tag{7.1}$$

$$\begin{aligned}
&\bar{\theta}\,(x,p)\,H\,(L-|x|) = \bar{F}_1(x,p)*\bar{U}_2^1(x,p) + \bar{F}_2(x,p)*\bar{U}_2^2(x,p)+\\
&+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k - \gamma\theta_k\right)\bar{U}_2^1(x+L,p) + \bar{u}_k\bar{U}_2^1{}_{,x}\,(x+L,p)\right\}+\\
&+H\,(t)\left\{\sum_{k=1}^{2}(-1)^{k+1}\left(\bar{q}_k - \eta p\bar{u}_k\right)\bar{U}_2^2\left(x-(-1)^kL,p\right) + \bar{\theta}_k\bar{U}_2^2{}_{,x}\,(x+L,p)\right\}
\end{aligned} \tag{7.2}$$

Here, a dash over a function indicates its Laplace transform.

Using the asymptotic properties of the fundamental matrix $\bar{U}_j^i$ at zero (6.2), from (7.1)-(7.2) we obtain the system of four linear equations at the boundary points to determine the Laplace transformants of unknown boundary functions, respectively to considered BVP. It has the following form:

$$\begin{aligned}
&0,5\bar{u}\,(-L,p) = \left(\bar{F}_1 \underset{x}{*} \bar{U}_1^1 + \bar{F}_2 \underset{x}{*} \bar{U}_1^2\right)_{x=L} +\\
&+c^2\sum_{k=1}^{2}(-1)^{k+1}\Big\{\left(\bar{p}_k(p) - \breve{\gamma}\,\bar{\theta}_k(p)\right)\bar{U}_1^1(-L-(-1)^kL,p)+\\
&\qquad +\bar{u}_k(p)\bar{U}_1^1{}_{,x}\,(-L-(-1)^kL,p)\Big\}+\\
&+\sum_{k=1}^{2}(-1)^{k+1}\Big\{\left(\bar{q}_k(p) + i\omega\eta\bar{u}_k(p)\right)\bar{U}_1^2\left(-L-(-1)^kL,p\right)+\\
&\qquad +\bar{\theta}_k(p)\bar{U}_1^2{}_{,x}\left(-L-(-1)^kL,p\right)\Big\}.
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
&-0,5\bar{u}\,(L,p) = \left(\bar{F}_1 \underset{x}{*} \bar{U}_1^1 + \bar{F}_2 \underset{x}{*} \bar{U}_1^2\right)_{x=L} +\\
&+c^2\sum_{k=1}^{2}(-1)^{k+1}\Big\{\left(\bar{p}_k(p) - \breve{\gamma}\bar{\theta}_k(p)\right)\bar{U}_1^1(L-(-1)^kL,p)+\\
&\qquad +\bar{u}_k(p)\bar{U}_1^1{}_{,x}\,(L-(-1)^kL,p)\Big\}+
\end{aligned} \tag{7.4}$$

$$+\sum_{k=1}^{2}(-1)^{k+1}\left\{(\bar{q}_k(p)-p\eta\bar{u}_k(\omega))\,\bar{U}_1^2\left(L-(-1)^kL,p\right)+\right.$$
$$\left.+\bar{\theta}_k(p)\bar{U}_1^2{}_{,x}\left(L-(-1)^kL,p\right)\right\}$$

$$0,5\bar{\theta}\left(-L,p\right)=\left.\left(\bar{F}_1\underset{x}{*}\bar{U}_2^1+\bar{F}_2\underset{x}{*}\bar{U}_2^2\right)\right|_{x=-L}+$$
$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k(p)-\gamma\bar{\theta}_k(p)\right)\bar{U}_2^1(0,p)+\bar{u}_k(p)\bar{U}_2^1{}_{,x}(0,p)\right\}+ \tag{7.5}$$
$$+\sum_{k=1}^{2}(-1)^{k+1}\left(\bar{q}_k(p)-p\eta\bar{u}_k(p)\right)\bar{U}_2^2\left(-L-(-1)^kL,p\right)+\bar{\theta}_k(\omega)\bar{U}_2^2{}_{,x}(0,p)$$

$$-0,5\bar{\theta}\left(L,p\right)=\left.\left(\bar{F}_1\underset{x}{*}\bar{U}_2^1+\bar{F}_2\underset{x}{*}\bar{U}_2^2\right)\right|_{x=L}+$$
$$+c^2\sum_{k=1}^{2}(-1)^{k+1}\left\{\left(\bar{p}_k(p)-\gamma\bar{\theta}_k(p)\right)\bar{U}_2^1(2L,p)+\bar{u}_k(p)\bar{U}_2^1{}_{,x}(2L,p)\right\}+ \tag{7.6}$$
$$+\sum_{k=1}^{2}(-1)^{k+1}\left(\bar{q}_k(p)-p\eta\bar{u}_k(p)\right)\bar{U}_2^2\left(L-(-1)^kL,p\right)+\bar{\theta}_k(p)\bar{U}_2^2{}_{,x}(2L,p).$$

From this system it is possible to obtain the resolving equations for any of the four BVPs.

8. Resolving equations of BVPs in Laplace transform space. The resolving system of linear algebraic equations (7.3)-(7.6) is represented in matrix form:

$$\{A1\}\times\left\{\begin{array}{c}\bar{u}_1\\ \bar{p}_1\\ \bar{\theta}_1\\ \bar{q}_1\end{array}\right\}+\{A2\}\left\{\begin{array}{c}\bar{u}_2\\ \bar{p}_2\\ \bar{\theta}_2\\ \bar{q}_2\end{array}\right\}=b \tag{8.1}$$

where

$$\{A1\}=$$
$$\left\{\begin{array}{cccc}0.5 & 0 & 0 & 0\\ -\left(c^2\bar{U}_1^1{}_{,x}-p\eta U_1^2\right)_{(2L)} & -c^2\bar{U}_1^1(2L,p) & \left(\widetilde{\gamma}c^2\bar{U}_1^1-\bar{U}_1^2{}_{,x}\right)_{(2L)} & -\bar{U}_1^2\left(2L,p\right)\\ 0 & 0 & 0.5 & 0\\ -\left(c^2\bar{U}_2^1{}_{,x}-p\eta\bar{U}_2^2\right)_{(2L)} & -c^2\bar{U}_2^1\left(2L,p\right) & \left(\widetilde{\gamma}c^2\bar{U}_2^1-\bar{U}_2^2{}_{,x}\right)_{(2L)} & -\bar{U}_2^2\left(2L,p\right)\end{array}\right\}$$

$$\{A2\}=$$
$$\left\{\begin{array}{cccc}\left(c^2\bar{U}_1^1{}_{,x}-p\eta\bar{U}_1^2\right)_{(-2L)} & c^2\bar{U}_1^1(-2L,p) & -\left(\widetilde{\gamma}c^2\bar{U}_1^1-\bar{U}_1^2{}_{,x}\right)_{(-2L)} & \bar{U}_1^2\left(-2L,p\right)\\ 0.5 & 0 & 0 & 0\\ \left(c^2\bar{U}_2^1{}_{,x}-p\eta U_2^2\right)_{(-2L)}, & c^2\bar{U}_2^1\left(-2L,p\right) & -\left(\widetilde{\gamma}c^2\bar{U}_2^1-\bar{U}_2^2{}_{,x}\right)_{(-2L)} & U_2^2\left(-2L,p\right)\\ 0 & 0 & 0.5 & 0\end{array}\right\}$$

$$b_1=(\bar{F}_1*\bar{U}_1^1+\bar{F}_2*\bar{U}_1^2)_{(-L},\quad b_2=(\bar{F}_1*\bar{U}_1^1+\bar{F}_2*\bar{U}_1^2)_{(L)},$$
$$b_3=(\bar{F}_1*\bar{U}_2^1++\bar{F}_2*\bar{U}_2^2)_{(-L)},\quad b_4=(\bar{F}_1*\bar{U}_2^1+\bar{F}_2*\bar{U}_2^2)_{(L)}$$

From this system it is necessary to construct a linear system of algebraic equations for any of the considered boundary value problems, leaving on the left side terms with unknown boundary values of the desired functions and transferring them to the right side with the known ones. The solution of this system is determined by use Cramer method.

After determining the missing boundary functions using formulas (7.1), (7.2), we determine the displacements and temperature in the rod. To determine thermoelastic stresses, we substitute the solution into the Duhamel-Neumann law (1.3), where all incoming functions are defined above. The obtained solutions make it possible to determine the thermally stressed state of bar structures with various geometric dimensions and thermoelastic parameters. In this case, one can study the effect of concentrated heat and power sources on them, described by singular generalized functions.

As example we present resolving boundary equations of BVP 1. In this case from (1.5) we know stresses and heat fluxes at rod ends, but (p_1, p_2, q_1, q_2) are unknowns. Then we obtain from (8.1) resolving system of equations (RES 1):

$$\{A\}\left\{\begin{array}{c}\bar{p}_1\\ \bar{q}_1\\ \bar{p}_2\\ \bar{q}_2\end{array}\right\} = b + \{B\}\times\left\{\begin{array}{c}\bar{u}_1\\ \bar{\theta}_1\\ \bar{u}_2\\ \vec{\theta}_2\end{array}\right\} \tag{8.2}$$

where the components of the matrices A, B are determined through the components of the matrices **A1** and **A2** as follows:

$$A=\left\{\begin{array}{cccc} A1_{12} & A1_{14} & A2_{12} & A2_{14}\\ A1_{22} & A1_{24} & A2_{22} & A2_{24}\\ A1_{32} & A1_{34} & A2_{32} & A2_{34}\\ A1_{42} & A1_{44} & A2_{42} & A2_{44}\end{array}\right\} \tag{8.3}$$

$$B=-\left\{\begin{array}{cccc} A1_{11} & A1_{13} & A2_{11} & A2_{13}\\ A1_{21} & A1_{23} & A2_{21} & A2_{23}\\ A1_{31} & A1_{33} & A2_{31} & A2_{33}\\ A1_{41} & A1_{43} & A2_{41} & A2_{43}\end{array}\right\} \tag{8.4}$$

$$b=\left\{\begin{array}{c} \left(F_1 \underset{x}{*} \bar{U}_1^1 + F_2 \underset{x}{*} \bar{U}_1^2\right)\big|_{x=-L}\\ \left(F_1 \underset{x}{*} \bar{U}_1^1 + F_2 \underset{x}{*} \bar{U}_1^2\right)\big|_{x=-L}\\ \left(F_1 \underset{x}{*} \bar{U}_2^1 + F_2 \underset{x}{*} \bar{U}_2^2\right)\big|_{x=L}\\ \left(F_1 \underset{x}{*} \bar{U}_2^1 + F_2 \underset{x}{*} \bar{U}_2^2\right)\big|_{x=L}\end{array}\right\} \tag{8.5}$$

9. Problems of periodic vibrations and their solutions. Periodic action of external vibration source is typical in practice. Their action can be presented in the form of Fourier series of stationary harmonic vibration which periods are multiply to base period. The solutions of such problems are determined also as Fourier series:

$$u\left(x,t\right)=\sum_j a_j e^{-i\omega_j t},\quad \theta\left(x,t\right)=\sum_j b_j e^{-i\omega_j t}$$

Then for every harmonic of this series we have stationary vibrations BVP by frequency ω_j. Using this method we can calculates thermo stress-state state of rod for every harmonics of this series and solve BVP. It gives possibility to investigate thermoelastic state of rods as at big oscillation periods and so at small periods, when uncoupled model of thermoelasticity is insufficient for application.

Let consider a rod fixed at the ends, whose temperature fluctuates with frequency ω at the ends

$$u\left(x_j,t\right)=0,\quad \theta(x_j,t)=\exp(-i\omega t);\quad j=1,2$$

It is BVP 1 with RES (8.2).

Figures 2,4,6 (a,b) show the amplitudes of displacements and temperature along the rod for different frequencies: $\omega = 0 : 1; 1; 10$. The calculations are performed for dimensionless parameters: $\gamma = 1, \eta = 1, \kappa = 1, c = 4$. In this case

In figures Fig. 3,5,7 (a,b) the real (green lines) and imaginary (blue lines) parts of complex amplitudes of displacements and temperature are depicted, which describe the displacements and temperature at fixed moments of time, spaced apart by a quarter of the oscillation period: : $t = 2\pi n/\omega(Ru, RT)$ and $t = 2\pi n/\omega + \pi/2\omega(Iu, IT), n = 0, 1, 2.$

The formation of standing thermoelastic waves has been observed. At low frequencies a middle of the rod is stationary, and maximum of longitudinal displacements are observed at quarter of length from rod ends. Maximum of temperature is in a middle of the rod. At low frequencies, temperature maximum in a middle of a rod is higher than temperature at its ends. With increasing frequency, a number of local extrema increases and temperature amplitudes increases in comparison with its value at rod ends.

The nodal points appear where both displacements and temperature are close or equal to zero. But extrema of amplitudes of displacement and temperatures are shifted relative to each other (where the displacements are zero, temperature amplitude maximum is observed).

10. Resonance vibrations of thermoelastic rod. One of the most important of engineering problems is to determine the spectrum of free vibrations of a thermoelastic rod (resonant frequencies). As you know, external influences at resonant frequencies often lead to devastating consequences for structures containing such elements.

To determine the spectrum of thermoelastic vibrations of the rod, one should study the determinant of RES matrix. Namely, the resonant frequencies must satisfy the characteristic equation

$$\det\left(\mathbf{A}(L,\omega_k)\right)=0,\qquad k=1,2...$$

This is a complex transcendental equation because the components of the fundamental matrix are expressed in terms of trigonometric functions of complex arguments. Its behavior and roots can be determined only numerically using various standard computer programs. For the system (8.2), the zeroes of determinant of matrix **A2** determine the resonant frequencies at which time-periodic solutions do not exist

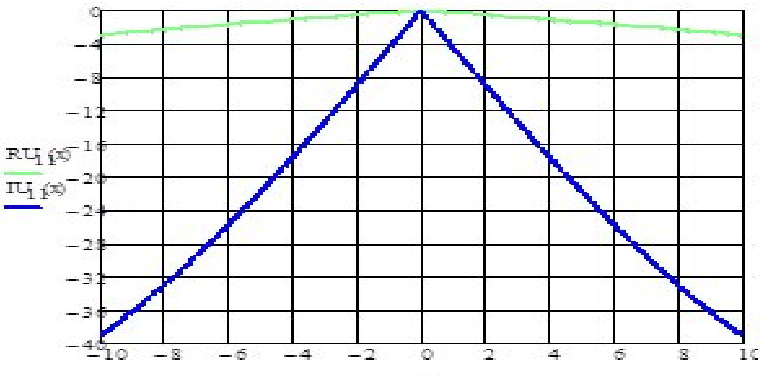
But in figure 9 there is graphs of determinants of matrices **A1** and **A2**. They are plotted, depending on the frequency ω. $Det(\mathbf{A2}\ (\omega))$ does not vanish anywhere. That is, in contrast to the dynamics of elastic rods, there are no classic resonant frequencies at which stationary periodic solutions do not exist. Such behavior of determinant of RES matrix is observed for all considered above BVPs.

However, there is a local minima on these curves. It shows that external action on such frequencies will cause increasing oscillation of rods, resonances in rod structures.

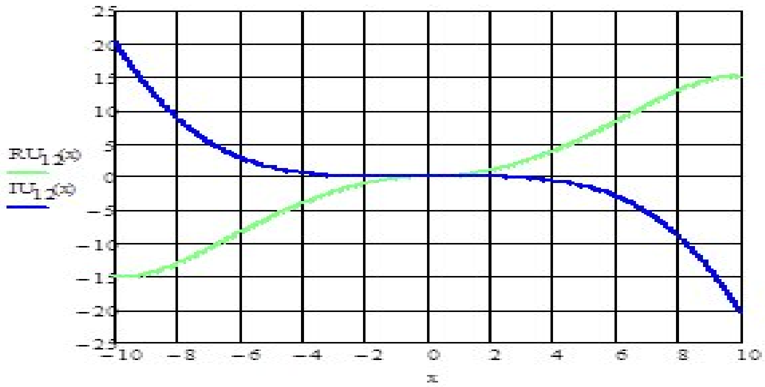
Table 1 presents the maximum amplitudes of displacements and temperature in the considered frequency range. With increasing frequency, the amplitude of the displacements increases sharply, and then begins to fall. The same is observed for temperature. With temperature fluctuations at the ends, the maximum amplitude of temperature fluctuations in the rod increased by 20 % .

Table 1

ω	U max	T max
0.1	0.0022	1.001
1	0.032	1.168
10	0.443	1.2
100	0.28	1.04

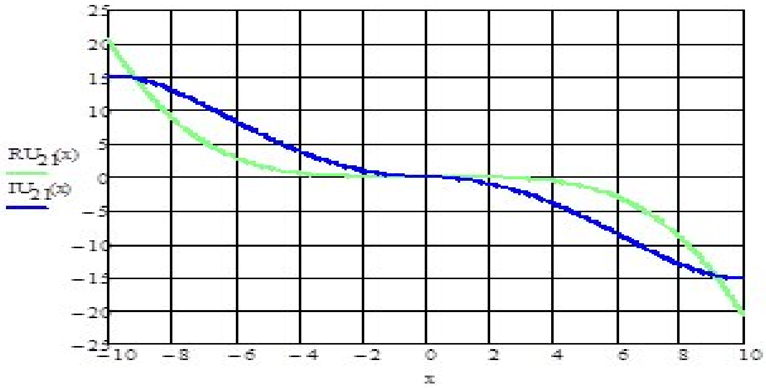


a

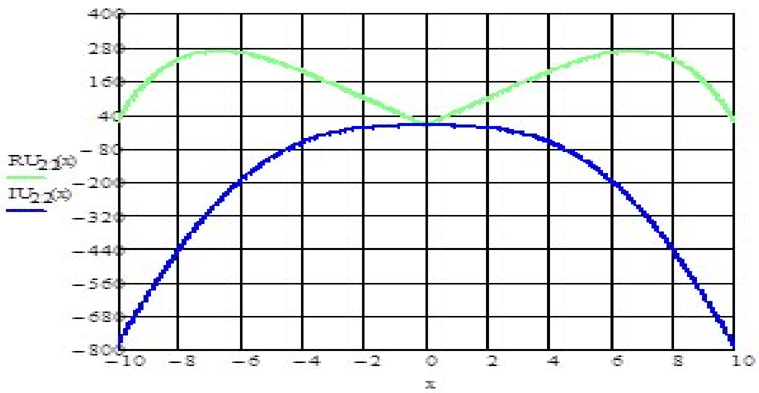


b

Figure 1. a,b - Components $\tilde{U}_k^j(x, w): w = 1 \quad (\gamma = 0.1,\ c = 1,\ k = 1,\ \eta = 1)$

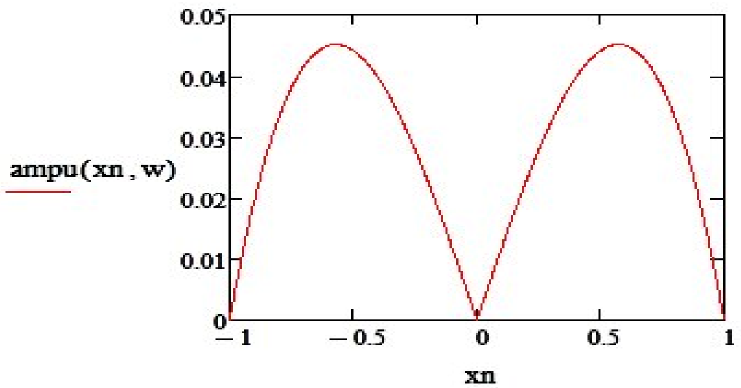


c

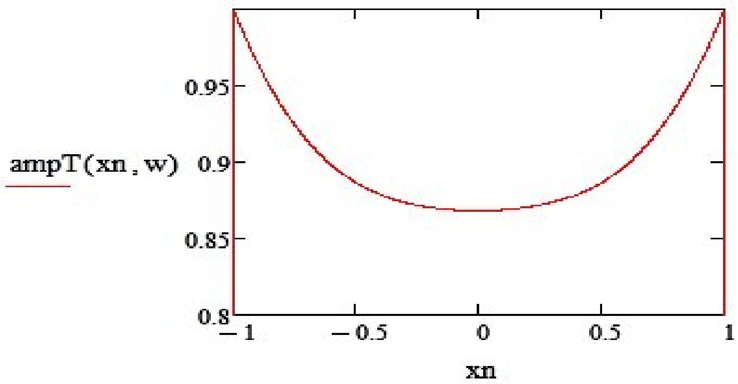


d

Figure 2. c, d - Component $\tilde{U}_k^j(x, w): w = 1 \quad (\gamma = 0.1,\ c = 1,\ k = 1,\ \eta = 1)$

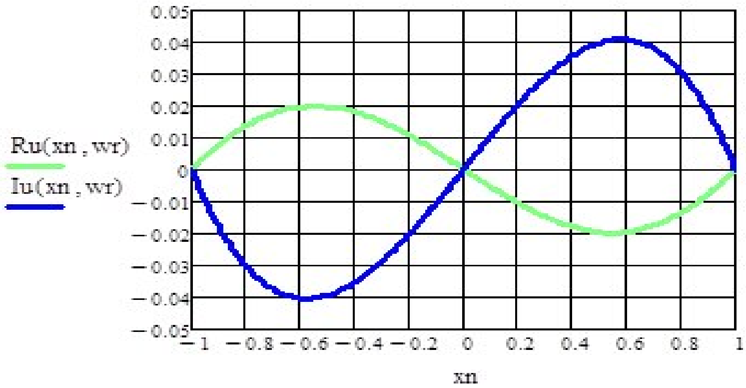


a

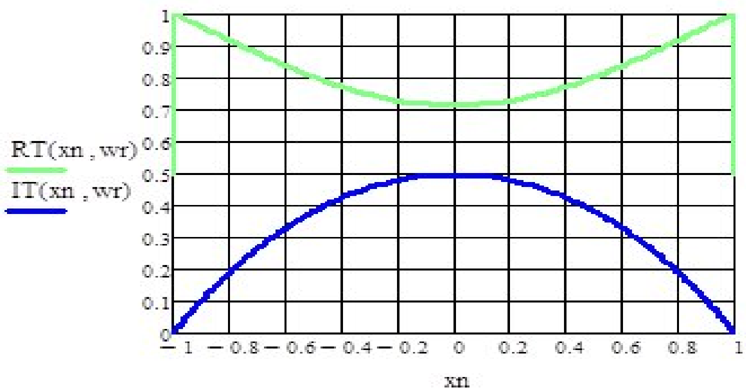


b

Figure 3. Amplitude (a) of displacements and their real and imaginary parts (b) along the rod: $\omega = 1$

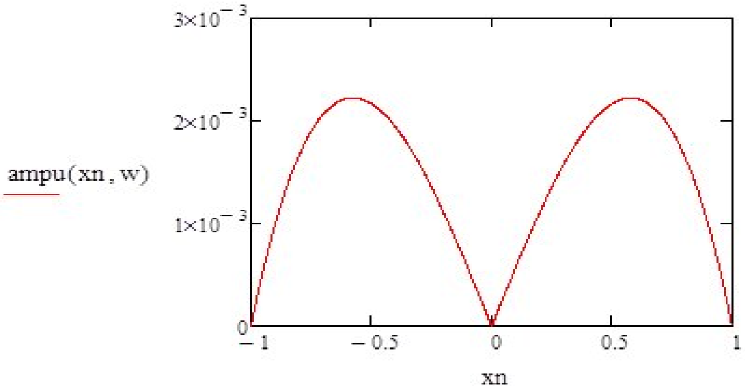


a

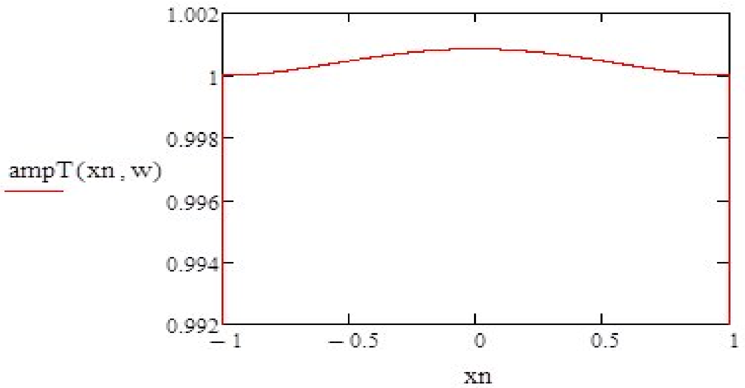


b

Figure 4. The amplitude (a) of temperature fluctuations and its real and imaginary part (b) along the rod: $\omega = 1$

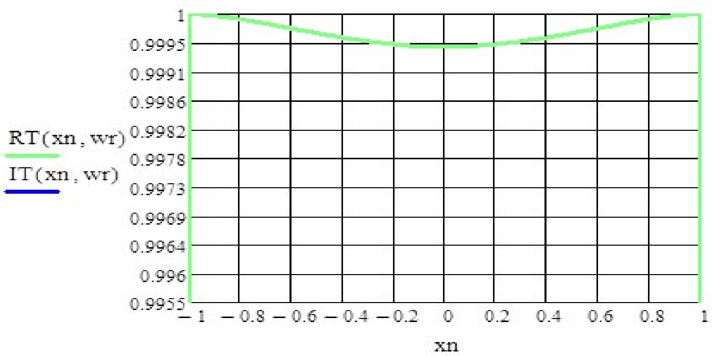


a

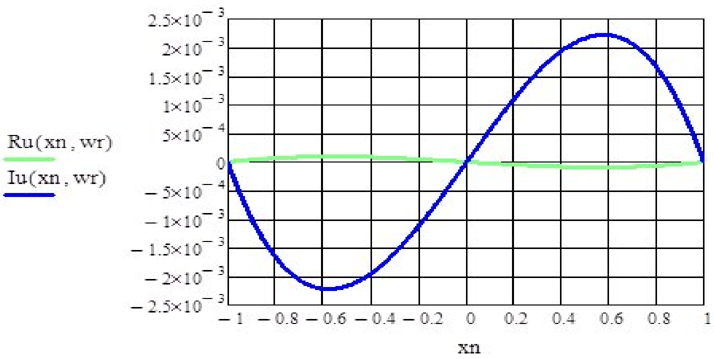


b

Figure 5. Amplitudes of displacements (a) and temperature (b) over shaft length: ω =0.1

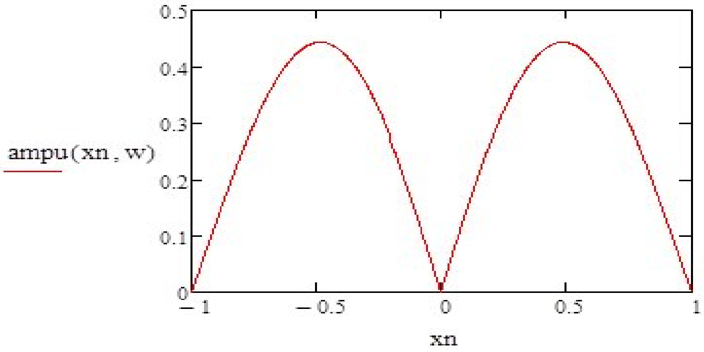


a

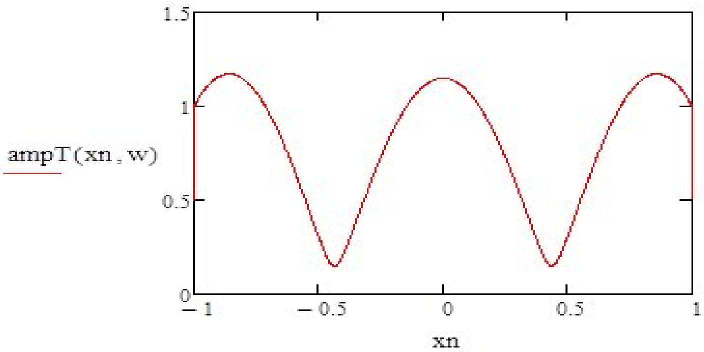


b

Figure 6. Displacements and temperature along the length of the rod at $t = 2\pi n/\omega$ Pë $t = 2\pi n/\omega + \pi/2\omega$: ω =0.1

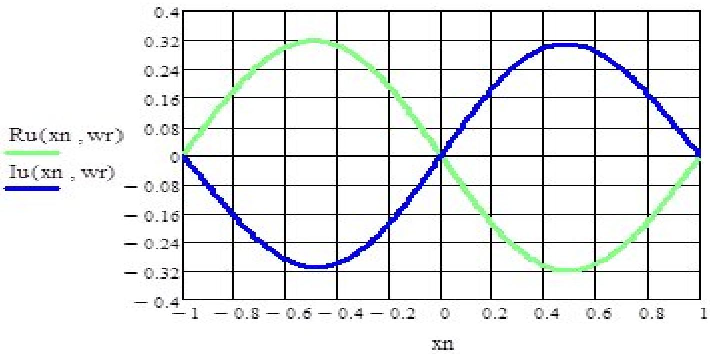


a

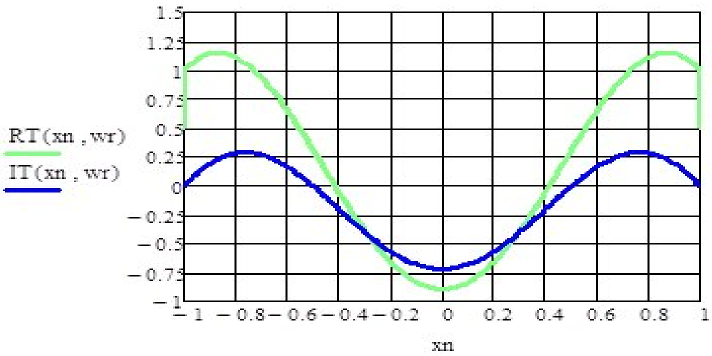


b

Figure 7. Amplitudes of displacements and temperature along the length of the rod: $\omega = 10$

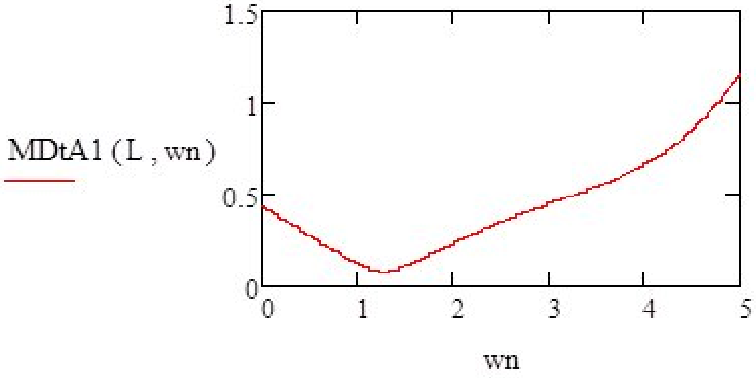


a

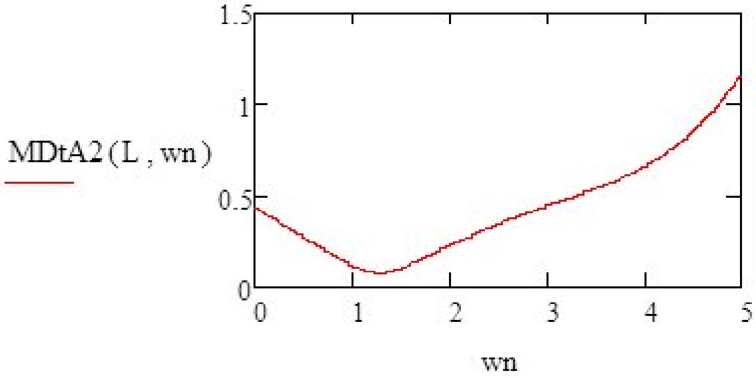


b

Figure 8. Displacements (a) and temperature (b) along the length of the rod at $t = 2\pi n/\omega$ Рё $t = 2\pi n/\omega + \pi/2\omega$: $\omega = 10$



a



b

Figure 9. Dependence of the determinants of the matrices A1 (a) and A2 (2) resolution system of equations of frequency

Conclusion. Obviously, we can consider combined problems with one type of boundary conditions at one end of a rod and other at second end, and other asymmetric conditions for a number of defined functions at rod ends. Constructed here Resulting Eguations System (8.1) gives possibility to solve 35 BVPs with different boundary conditions by different external thermal and forces action. It needs to set 4 boundary function from 8. Then others 4 are defined from RES (8.1). You can know 2 arbitrary boundary function from 4 at both rod ends, or 3 from 4 at one end and 1 any boundary function at other ends, or all 4 only at one ends.

Also by $\gamma = 0$ this system describes the dynamics of elastic rods neglecting thermal stresses but with a glance of velocity of deformation on its temperature.

Note also that formulae (7.1)-(7.2) may be applied for engineering calculations of rods constructions for estimating their durability and safety without construction RES and its solving.

This work was supported by the Grant of Ministry of Education and Science of the Republic of Kazakhstan No. AP05132272 “Boundary value problems of the dynamics of deformable solid and electromagnetic media and their solving”.

References

1 Новацкий В. Теория упругости. - Москва: "Мир", 1975.

2 Новацкий В. Динамические задачи термоупругости. Москва: "Мир", 1970. - 256 с.

3 Купрадзе В.Д. , Гегелиа Т.Г., Башелейшвили М.О., Бурчуладзе Т.В. Трехмерные задачи математической теории упругости и термоупругости.-Москва: Наука. -1976. - 664 с.

4 Fleurier J., Predeleanu M. On the use of coupled fundamental solutions in boundary element method for thermoelastic problems // Eng. Anal. -1987. - V.4. - N2. - P. 70-74.

5 Sah J., Tasaka N. Boundary element analysis of linear coupled thermoelasticity problems by using Laplace transformation // Boundary Elem. Meth. Appl. Mech. Proc 1st Joint Jap., US Symp. Boundary Elem. Meth. - Tokyo, 3-6 Oct. - 1988. -P. 335-544.

6 Sharp S., Crouch S.L. Boundary integral methods for thermoelasticity problems //Trans. ASME: J. Appl. Mech. -1986. -Vol. 53. № 2. -P.298-302.
7 Алексеева Л.А., Купесова Б.Н. Метод обобщенных функций в краевых задачах связанной термоэластодинамики // Прикладная математика и механика. - 2001. - Т.65 . - №2. - С.334-345.
8 Алексеева Л.А. Метод обобщенных функций в нестационарных краевых задачах для волнового уравнения // Математический журнал. - 2006. - Т.6. - №1(19). - С.16-32.
9 Владимиров В.С. Обобщенные функции в математической физике. Москва: Наука, 1978.
10 Кеч В., Теодореску П. Введение в теорию обобщенных функций с приложениями в технике. Москва: "Мир", 1978. -518 с.
11 Алексеева Л .А., Ахметжанова М.М. Фундаментальные и обобщенные решения уравнений динамики термоупругих стержней. 1 Стационарные колебания // Математический журнал. - 2014. - Т.14. - №2(52). - С.5-20.
12 Алексеева Л.А, Дадаева А.Н., Айнакеева Н.Ж. Фундаментальные и обобщенные решения уравнений нестационарной динамики термоупругих стержней // Вестник ЕНУ им. Л. Гумилева. -2018.-№2(123). - С.56-65.

Л.А. Алексеева, М.М. Ахметжанова

Математика және математикалық моделдеу институты, Алматы, Қазақстан

Термосерпімді өзек динамикасының шектік есептерінің жалпылама функциялар әдісі

Аннотация: Байланысқан термосерпімді моделін пайдалана отырып, термосерпімді өзек динамикасының стационарлық емес және тербелісті шектік есептерінін шешу үшін жалпалама функциялар әдісі әзірленді. Соққы жүктемелері мен жылу ағындары әсерінен құрылымдарда пайда болатын термосерпімді соққы толқындары қарастырылды. Олардың бағыттарының шарттары алынды. Соққы толқындарын ескере шектік есептерінің жалғыздығы дәлелденді. Шектік есептерінің кең класстар аналитикалық шешімдерін анықтау үшін МОФ негізінде олардың алгебралық шешу теңдеулер жүйесі құрастырылды. Түрлі типтегі жылу көздерінің және күш әсерінің өзек динамикасы зерттеледі, соңын ішінде сингулярлы жалпылама функцияларын сипатайтын, импульсті концентрацияланған көздер әсерін моделдеуге мүмкіндік береді. Стационарлы тербеліс кезінде бір шектік есептер шешімін компьютерлік іске асыру жүргізіледі, төменгі және жоғарғы жиіліктердегі өзек термодинамикасын есептеудің сандық эксперименттерінің нәтижелері көрсетілді. Осы шешімдер мен алгоритмдер өзек құрылымдарының беріктік қасиеттерін бағалау үшін олардың инженерлік есептеулер үшін қолдануы мүмкін.
Түйін сөздер: термопластика, өзек, шектік есеп, кернеулі - деформациялық күй, жалпылама функциялар әдісі.

Л.А. Алексеева, М.М. Ахметжанова

Институт математики и математического моделирования, Алматы, Казахстан

Метод обобщенных функций в краевых задачах динамики термоупругого стержня

Аннотация: Разработан метод общих функций (МОФ) для решения нестационарных и вибрационных краевых задач динамики термоупругого стержня с использованием модели связанной термоупругости. Рассмотрены термоупругие ударные волны, возникающие в таких конструкциях под действием ударных нагрузок и тепловых потоков. Получены условия на их фронтах. Доказана единственность поставленных краевых задач с учетом ударных волн. На основе МОФ построена система алгебраических разрешающих уравнений для широкого класса краевых задач для определения их аналитических решений. Исследуется динамика стержня под действием сил и источников тепла различного типа, в том числе описываемых сингулярными обобщенными функциями, которые позволяют моделировать воздействие импульсных концентрированных источников. Проведена компьютерная реализация решений одной краевой задачи при стационарных колебаниях, приведены результаты численных экспериментов расчета термодинамики стержня на низких и высоких частотах. Данные решения и алгоритмы могут быть применены для инженерных расчетов стержневых конструкций для оценки их прочностных свойств.
Ключевые слова: термопластичность, стержень, краевые задачи, напряженно-деформированное состояние, метод общих функций.

References

1 Novatsky V. Teoriya uprugosti [Theory of Elasticity] (Mir, Moscow, 1975).
2 Novatsky V. Dinamicheskie zadachi termouprugosti [Dynamic problems of thermoelasticity] (Mir, Moscow, 1970).
3 Kupradze V.D., Gegelia T.G., Bacheleshvili M.O., Burchuladze T.V. Trekhmernye zadachi matematicheskoj teorii uprugosti i termouprugosti [Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity] (Nauka, Moscow, 1976, 663 p.).
4 Fleurier J., Predeleanu M. On the use of coupled fundamental solutions in boundary element method for thermoelastic problems, Engin. Anal. 1987. Vol. 4. № 2. P. 70-74.
5 Sah J., Tasaka N. Boundary element analysis of linear coupled thermoelasticity problems by using Laplace transformation /Boundary Elem. Meth. Appl. Mech. Proc 1st Joint Jap., US Symp. Boundary Elem. Meth., Tokyo, 3-6 Oct, 1988, pp. 335-544.

6 Sharp S., Crouch S.L. Boundary integral methods for thermoelasticity problems.Trans, ASME: J. Appl. Mech. 1986. Vol.53. № 2. P. 298-302.
7 Alexeyeva L.A., Kupesova B.N. Metod obobshchennyh funkcij v kraevyh zadachah svyazannoj termoelastodinamiki[The method of generalized functions in boundary value problems related thermoelastodynamics], Prikladnaya matematika i mekhanika [Applied Mathematics and Mechanics], 2001. Vol. 65. № 2. P. 334-345.
8 Alexeyeva L.A. Metod obobshchennyh funkcij v nestacionarnyh kraevyh zadachah dlya volnovogo uravneniya[The method of generalized functions in non-stationary boundary value problems for the wave equation], Matematicheskij zhurnal[Matematical Journal], 2006. Vol. 6. № 1. P. 16-32.
9 Vladimirov V.S. Obobshchennye funkcii v matematicheskoj fizike[Generalized functions in mathematical physics] (Nauka, Moscow, 1978).
10 Kech V., Teodoresku P. Vvedenie v teoriyu obobshchennyh funkcij s prilozheniyami v tekhnike [Introduction to generalized functions theory with applications in technics] (Mir, Moscow, 1978).
11 Alexeyeva L.A., Akhmetzhanova M.M. Fundamental'nye i obobshchennye resheniya uravnenij dinamiki termouprugih sterzhnej. 1 Stacionarnye kolebaniya [Fundamental and generalized solutions of the equations of dynamics of thermoelastic cores. 1. Stationary fluctuations], Matematicheskij zhurnal [Mathematical journal], 2014. Vol. 14. № 2. P. 5-20.
12 Alexeyeva L.A., Dadaeva A.N., Ainikeeva N.Zh. Fundamental'nye i obobshchennye resheniya uravnenij nestacionarnoj dinamiki termouprugih sterzhnej [Fundamental and generalized solutions of the equations of the non-stationary dynamics of thermoelastic rods], Bulletin of L.N. Gumilyov Eurasian National University. Mathematics. Computer Sciences. Mechanics Series 2018. Vol. 123. № 2. P. 56-65.

Information about author:

Алексеева Л.А. - д.ф.м.н., проф., заведующая Отделом математической физики и моделирования ИМММ МОН РК, Алматы, Казахстан.

Ахметжанова М.М. - Научный сотрудник, Институт математики и математического моделирования, Алматы, Казахстан

Alekseeva L.A. - Doctor of Phys.-Math. Sciences, Head of the Department of Mathematical Physics and Modeling of the Institute Of Mathematics And Mathematical Modeling Ministry of Education and Science Republic of Kazakhstan, Almaty, Kazakhstan.

Akhmetzhanova M.M. - Research fellow, Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan.

Поступила в редакцию 11.02.2020