

Article
IRSTI: 27.39.19

CHARACTERIZATION OF THE BOUNDEDNESS OF THE DISCRETE FRACTIONAL MAXIMAL OPERATOR BETWEEN WEIGHTED LORENTZ SEQUENCE SPACES¹

A.Gogatishvili¹, N.A.Bokayev², N.K.Kuzeubayeva^{2*}

¹ Institute of Mathematical of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

² L.N. Gumilyov Eurasian National University, Satpaev Str. 2, 010008, Astana, Republic of Kazakhstan

(E-mail: ¹ gogatish@math.cas.cz, ² bokayev2011@yandex.ru, ² *nurgul.kuzeubaeva@mail.ru)

Abstract. In this paper, we give the complete characterization of the boundedness of the discrete fractional maximal operator M_γ , defined on $\mathbb{Z}^n$, $n \in \mathbb{N}$, $\gamma \in [0, n)$, between the classical Lorentz sequence spaces $\lambda_p(v)$ and $\lambda_q(w)$, where $0 < p, q < \infty$, $\{v(m)\}$ and $\{w(m)\}$ are non-negative sequences defined on $\mathbb{N}$. We first obtain a sharp upper estimate for the nonincreasing rearrangement of the discrete fractional maximal operator $M_\gamma x$, with a supremal operator involving a discrete Hardy operator and the nonincreasing rearrangement of the sequence $\{x(k)\}$, and we show that this estimate is sharp by showing a lower estimate for special sequences. Using this result, we can reduce the characterization of the boundedness of the discrete fractional maximal operator M_γ between the discrete weighted Lorentz sequence spaces to the boundedness of the discrete supremum operator between weighted Lebesgue sequence spaces, restricted to nonincreasing sequences. This problem was investigated in the recent paper by the authors, which enables us to obtain the final characterization.

Keywords: discrete Fractional maximal operator, weighted Lorentz sequence space, supremal operator, Hardy operator.

DOI: <https://doi.org/10.32523/bulmathenu.2026/2.1>

2000 Mathematics Subject Classification: 46B30, 46B85

1. Introduction

We start with the definition of the decreasing rearrangement of the real sequences. We will follow the book [1] reformulating the definitions for the special case of the σ -finite measure space

¹The research was supported by the grant Ministry of Education and Science of the Republic of Kazakhstan (project no. AP26196065). The research of A. Gogatishvili was partially supported by the grant project 23-04720S of the Czech Science Foundation (GAČR), The Institute of Mathematics, CAS is supported by RVO:67985840, by Shota Rustaveli National Science Foundation (SRNSF), grant no: FR22-17770.

$(\mathbb{Z}^n, \mathcal{P}(\mathbb{Z}^n), m)$ where m is the usual counting measure, which is defined as follows: for a finite set A , $m(A)$ is the number of its elements, and for infinite A we have that $m(A) = \infty$.

Definition 1 (Distribution function and decreasing rearrangement). *Consider the measure space $(\mathbb{Z}^n, \mathcal{P}(\mathbb{Z}^n), m)$. For any bounded sequence $\{x(k)\}_{k \in \mathbb{Z}^n}$ of real numbers, we define its distribution function $m_x : [0, \infty) \rightarrow [0, \infty]$ in the following way:*

$$m_x(\omega) = m\{k \in \mathbb{Z}^n : |x(k)| > \omega\}, \quad \omega > 0.$$

Then for $t \geq 0$ we define the decreasing rearrangement:

$$x^*(t) = \inf\{\omega > 0 : m_x(\omega) \leq t\}$$

and the decreasing rearrangement $\{x^*(i)\}_{i=1}^\infty$ of the sequence $\{x(k)\}_{k \in \mathbb{Z}^n}$ as

$$x^*(i) = x^*(t(i)) = \inf\{\omega > 0 : m_x(\omega) \leq t(i)\}$$

for any $i - 1 \leq t(i) < i$, $i \in \mathbb{N}$.

It should be noticed that an alternative equivalent definition of the decreasing rearrangement is possible.

$$x^*(i) = \sup_{E \subset \mathbb{Z}^n, m(E)=i} \min_{k \in E} |x(k)| \quad i \in \mathbb{N}.$$

We use the convention here that $\inf \emptyset = \infty$. Thus, if $m_x(\omega) > i$ for all $\omega \geq 0$, then $x^*(i) = \infty$.

We denote $x^{**}(k)$ as the average $\{x^*(k)\}_{k \in \mathbb{N}}$:

$$x^{**}(k) = \frac{1}{k} \sum_{i=1}^k x^*(i), \quad k \in \mathbb{N}.$$

We will use the following conventions. We write $A \lesssim B$ if there exists a constant $C \in (0, \infty)$ independent of appropriate quantities such that $A \leq CB$. We write $A \approx B$ if we have both $A \lesssim B$ and $B \lesssim A$. We put $1/(\pm\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$. Moreover, we omit arguments of integrands as well as differentials in integrals when appropriate in order to keep the exposition as short as possible. We will denote by $LHS(*)$ and $RHS(*)$, the left-hand side and right-hand side of an inequality $(*)$, respectively. Note that since the counting measure m can only take integer values (or infinity), it does not matter which exact t_k is chosen for the given k , as long as it is in the aforementioned interval.

In this paper, an upper bound is obtained for a nonincreasing rearrangement of a discrete fractional-maximal operator through a certain suprema operator. We assume throughout that $\mathbf{w}$ and $\mathbf{v}$ are sequences of nonnegative terms. Their partial sums are denoted by

$$W(m) = \sum_{k=1}^m w(k), \quad V(m) = \sum_{k=1}^m v(k), \quad m \in \mathbb{N}.$$

In the case where the series converges, or we consider the limit of the sequence of partial sums, we write:

$$V(\infty) = \lim_{m \rightarrow \infty} V(m).$$

Definition 2. *Given a fixed sequence $w(k)_{k \in \mathbb{N}}$ of nonnegative real numbers not identically equal to zero and a fixed $p \in (0, \infty)$. The classical Lorentz sequence space $\lambda_p(w)$ is the set of all complex sequences $x = x(k)_{k \in \mathbb{Z}^n}$ such that $\|x\|_{\lambda_p(w)} < \infty$, where*

$$\|x\|_{\lambda_p(w)} = \begin{cases} \left(\sum_{k=1}^\infty (x^*(k))^p \cdot w(k) \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup_{k \geq 1} x^*(k) \cdot w(k), & p = \infty. \end{cases}$$

Notice that the $\|\cdot\|_{\lambda_p(w)}$ does not necessarily have to be a norm.

Definition 3.

$$\lambda_p(w) = \{x \in \mathbb{Z}^n : \|x\|_{\lambda_p(w)} < +\infty\}.$$

For $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n$ and $N \in \mathbb{N}$ we define the set

$$S_{m,N} := \{k \in \mathbb{Z}^n : \|k - m\|_\infty \leq N\},$$

where $\|(x(1), x(2), \dots, x(n))\|_\infty := \max\{|x(i)| : 1 \leq i \leq n\}$ for each $(x(1), x(2), \dots, x(n)) \in \mathbb{Z}^n$. Then $|S_{m,N}| = (2N+1)^n$ is the cardinality of the set $S_{m,N}$. Let $\gamma \in [0, n)$, $x = (x(1), x(2), \dots, x(k)) \in \mathbb{Z}^n$. The discrete fractional maximal operator M_γ is defined by

$$(M_\gamma x)(m) = \sup_{N \in \mathbb{N}} |S_{m,N}|^{\frac{\gamma}{n}-1} \sum_{k \in S_{m,N}} |x(k)|, \quad m \in \mathbb{Z}^n.$$

The aim of this paper is to establish a condition for the boundedness of the discrete fractional maximal operator from $\Lambda_p(v)$ to $\Lambda_q(w)$.

2. Preliminaries

Theorem 1. [1, Chapter-2, Theorem 2.2] *If $x = \{x(m)\}_{m \in \mathbb{Z}^n}$ and $b = \{b(m)\}_{m \in \mathbb{Z}^n}$ are complex sequences, then*

$$\sum_{m \in \mathbb{Z}^n} x(m)b(m) \leq \sum_{k=1}^{\infty} x^*(k)b^*(k).$$

Lemma 1. *Let $n \in \mathbb{N}$ and $\gamma \in [0, n)$, then the following inequalities hold:*

$$\sup_{k \in \mathbb{N}} (M_\gamma x)^*(k) \leq D_3 \sup_{k \in \mathbb{N}} k^{\frac{\gamma}{n}} x^*(k), \quad (1)$$

$$\sup_{k \in \mathbb{N}} k^{1-\frac{\gamma}{n}} (M_\gamma x)^*(k) \leq D_4 \sum_{k \in \mathbb{Z}^n} |x(k)|, \quad (2)$$

where D_3 and D_4 are positive constants depending only on n and γ .

Proof. Using the definition of M_γ and Theorem 1 we get

$$\begin{aligned} \sup_{k \in \mathbb{N}} (M_\gamma x)^*(k) &= \sup_{m \in \mathbb{Z}^n} (M_\gamma x)(m) \\ &= \sup_{m \in \mathbb{Z}^n} \sup_{N \in \mathbb{N}} |S_{m,N}|^{\frac{\gamma}{n}-1} \sum_{k \in S_{m,N}} |x(k)| \\ &\leq \sup_{m \in \mathbb{Z}^n} \sup_{N \in \mathbb{N}} |S_{m,N}|^{\frac{\gamma}{n}-1} \sum_{i=1}^{|S_{m,N}|} x^*(i) \\ &= \sup_{m \in \mathbb{Z}^n} \sup_{N \in \mathbb{N}} |S_{m,N}|^{\frac{\gamma}{n}-1} \sum_{i=1}^{|S_{m,N}|} i^{-\gamma/n} i^{\gamma/n} x^*(i) \\ &\leq \sup_{m \in \mathbb{Z}^n} \sup_{N \in \mathbb{N}} |S_{m,N}|^{\frac{\gamma}{n}-1} \sum_{i=1}^{|S_{m,N}|} i^{-\gamma/n} \sup_{i \in \mathbb{N}} i^{\gamma/n} x^*(i) \\ &\leq D_3 \sup_{i \in \mathbb{N}} i^{\gamma/n} x^*(i) \end{aligned}$$

the estimate (1) proved.

Now we prove the estimate (2). Fix a number $\lambda > 0$ and set

$$E_\lambda = \{k \in \mathbb{Z}^n : (M_\gamma x)(k) > \lambda\}.$$

We want to estimate

$$m(E_\lambda)^{1-\frac{\gamma}{n}} = m(\{k \in \mathbb{Z}^n : (M_\gamma x)(k) > \lambda\}) \leq \frac{\tilde{C}_4}{\lambda} \sum_{k \in \mathbb{Z}^n} |x(k)|.$$

For each $k \in E_\lambda$, choose N_k such that

$$\frac{1}{|S_{k,N_k}|^{1-\gamma/n}} \sum_{m \in S_{k,N_k}} |x(m)| > \lambda.$$

Thus $E_\lambda \subset \cup_{k \in E_\lambda} S_{k,N_k}$. Take any finite set $E \subset E_\lambda$, and apply the Vitali covering lemma [2, Chapter-1, Lemma, p. 9]. So there exists a disjoint subcollection $\{S_j = S_{k_j, N_{k_j}}\}$ selected by the Vitali covering lemma, such that:

$$m(E_\lambda) \leq 3^n \sum_j m(S_j)$$

and

$$\begin{aligned} m(E_\lambda)^{1-\frac{\gamma}{n}} &\leq 3^{n-\gamma} \sum_j m(S_j)^{1-\frac{\gamma}{n}} \\ &\leq 3^{n-\gamma} \lambda^{-1} \sum_j \sum_{m \in S_j} |x(m)| \\ &\leq 3^{n-\gamma} \lambda^{-1} \sum_{m \in \cup_j S_j} |x(m)| \\ &\leq 3^{n-\gamma} \lambda^{-1} \sum_{m \in \mathbb{Z}^n} |x(m)|, \end{aligned}$$

so

$$\begin{aligned} \sup_{\lambda > 0} \lambda m(E_\lambda)^{1-\frac{\gamma}{n}} &\leq 3^{n-\gamma} \sum_{m \in \mathbb{Z}^n} |x(m)| \\ \sup_{m \in \mathbb{N}} m^{1-\frac{\gamma}{n}} (M_\gamma x)^*(m) &= \sup_{\lambda > 0} \lambda m(E_\lambda)^{1-\frac{\gamma}{n}} \leq 3^{n-\gamma} \sum_{m \in \mathbb{Z}^n} |x(m)| \end{aligned}$$

which is an estimate (2) with $D_4 = 3^{n-\gamma}$. □

3. Main results

In the following theorem, an upper bound estimate for the nonincreasing rearrangement of the discrete fractional maximal operator is obtained.

Having defined the $\lambda_p(w)$ norm, we can now move on to the sequence spaces induced by that norm. For the definition to make sense, we only take sequences with finite norm.

Theorem 2. *Let $n \in \mathbb{N}$ and $\gamma \in [0, n)$. Then there exists a positive constant D_1 , depending only on n and γ , such that*

$$(M_\gamma x)^*(m) \leq D_1 \sup_{m \leq k < \infty} k^{\gamma/n} x^{**}(k), \quad m \in \mathbb{N} \quad (3)$$

for any sequence $x = \{x(m)\}_{m \in \mathbb{Z}^n}$.

Proof of Theorem 2. Fix $k \in \mathbb{N}$ and let $x \in l_{loc}^1(\mathbb{Z}^n)$. We may assume that

$$\sup_{m \leq k < \infty} k^{\gamma/n-1} \sum_{j=1}^k x^*(j) < \infty \quad (4)$$

otherwise (3) holds trivially. Then by the Hardy-Littlewood inequality (1)

$$\sum_{k \in Q} |x(k)| \leq \sum_{i=1}^k x^*(i) < \infty$$

for every set $Q \subset \mathbb{Z}^n$ of measure at most k . In particular, if we put

$$E = \{m \in \mathbb{Z}^n : |x(m)| > x^*(k)\},$$

then $m(E) \leq k$ [1, Chapter 2, (1.18)], and so $\sum_{k \in E} |x(k)| < \infty$. We define the sequences:

$$y_k(m) = \max\{|x(m)| - x^*(k), 0\} \operatorname{sgn} x(m), \quad m \in \mathbb{Z}^n, \quad (5)$$

$$z_k(m) = \min\{|x(m)|, x^*(k)\} \operatorname{sgn} x(m), \quad m \in \mathbb{Z}^n. \quad (6)$$

Then $x = y_k + z_k$ and

$$y_k^*(i) = \chi_{(0;k) \cap \mathbb{N}}(x^*(i) - x^*(k)), \quad i \in \mathbb{N} \quad (7)$$

$$z_k^*(i) = \min\{x^*(i), x^*(k)\}, \quad i \in \mathbb{N} \quad (8)$$

whence from (7):

$$\sum_{m \in \mathbb{Z}^n} |y_k(m)| = \sum_{i=1}^{\infty} y_k^*(i) = \sum_{i=1}^k (x^*(i) - x^*(k)) \leq \sum_{i=1}^k x^*(i). \quad (9)$$

From the inequality (2) it follows that

$$(M_\gamma y_k)^*(i) \leq D_4 i^{\gamma/n-1} \sum_{m \in \mathbb{Z}^n} |y_k(m)| \leq D_4 i^{\gamma/n-1} \sum_{i=1}^k x^*(i), \quad i \in \mathbb{N}. \quad (10)$$

By (1) and (8) we have:

$$\begin{aligned} (M_\gamma z_k)^*(i) &\leq D_3 \sup_{1 \leq i < \infty} i^{\gamma/n} z_k^*(i) \\ &= D_3 \max \left\{ \sup_{0 < i < k} i^{\gamma/n} x^*(k), \sup_{k \leq i < \infty} i^{\gamma/n} x^*(i) \right\} \\ &\approx D_3 \max \left\{ k^{\gamma/n} x^*(k), \sup_{k \leq i < \infty} i^{\gamma/n} x^*(i) \right\} \\ &\leq D_3 \sup_{k \leq i < \infty} i^{\gamma/n} x^*(i). \end{aligned}$$

Hence

$$\sup_{0 < i < \infty} (M_\gamma z_k)^*(i) \leq D_3 \sup_{k \leq i < \infty} i^{\gamma/n} x^*(i), \quad (11)$$

$$(M_\gamma x)^*(k) \leq (M_\gamma y_k)^*(k) + \sup_{0 < i < \infty} (M_\gamma z_k)^*(i).$$

By using, (10) and (11) we get

$$\begin{aligned} (M_\gamma x)^*(k) &\leq D_4 k^{\gamma/n-1} \sum_{i=1}^k x^*(i) + D_3 \sup_{k \leq i < \infty} i^{\gamma/n} x^*(i) \\ &\leq (D_4 + D_3) \sup_{k \leq i < \infty} i^{\gamma/n} x^{**}(i) \\ &= D_1 \sup_{k \leq i < \infty} i^{\gamma/n} x^{**}(i), \end{aligned}$$

with $D_1 = D_3 + D_4$. □

Theorem 3. Let us $n \in \mathbb{N}$ and $\gamma \in [0, n)$. Inequality (3) in Theorem 2 is sharp in the sense that for any positive nonincreasing sequence $x = \{x(m)\}_{m \in \mathbb{N}}$ there exists a sequence $y = \{y(k)\}_{k \in \mathbb{Z}^n}$ such that $y^*(k) = x(k)$, $k \in \mathbb{N}$ and the inequality holds:

$$\begin{aligned} (M_\gamma y)^*(m) &\geq D_2 \sup_{m \leq k < \infty} k^{\gamma/n} y^{**}(k) \\ &= D_2 \sup_{m \leq k < \infty} k^{\frac{\gamma}{n}-1} \sum_{i=1}^k x(i), \quad m, k \in \mathbb{N}, \end{aligned} \quad (12)$$

where D_2 - is a positive constant depending only on n and γ .

Proof of Theorem 3. Let for any $m \in \mathbb{N}$ define

$$N_1 = \{1\},$$

$$N_m := \{i \in \mathbb{N} : i = (2m-3)^n + 1, \dots, (2m-1)^n\}, \quad m = 2, 3, \dots,$$

and

$$Q_m := \{\bar{k} \in \mathbb{Z}^n : \max_{1 \leq i \leq n} |k(i)| = m-1\}.$$

It is clear that

$$\#\{N_1\} = \#\{Q_1\} = 1,$$

and

$$\#\{N_m\} = \#\{Q_m\} = (2m-1)^n - (2m-3)^n, \quad m = 2, 3, \dots$$

$$\mathbb{N} = \cup_{m \in \mathbb{N}} N_m, \quad \mathbb{Z}^n = \cup_{m \in \mathbb{N}} Q_m.$$

Let

$$\Theta_m : Q_m \rightarrow N_m, \quad m \in \mathbb{N}$$

be one-to-one functions.

Define $\Theta : \mathbb{Z}^n \rightarrow \mathbb{N}$

$$\Theta(k) = \Theta_m(k), \quad \text{if } k \in Q_m.$$

$$y(k) = x(\Theta(k)), \quad k \in \mathbb{Z}^n.$$

Then, for every $k, m \in \mathbb{Z}^n$ such that $\|k\|_\infty \geq \|m\|_\infty$, we have

$$\begin{aligned} M_\gamma(y)(m) &\geq c |S_{0, \|k\|_\infty}|^{\frac{\gamma}{n}-1} \sum_{j \in S_{0, \|k\|_\infty}} |y(j)| \\ &= c \left(2\|k\|_\infty + 1\right)^{\gamma-n} \sum_{i=1}^{\|k\|_\infty} \sum_{j \in Q_i} |y(j)| \\ &= c \left(2\|k\|_\infty + 1\right)^{\gamma-n} \sum_{i=1}^{\|k\|_\infty} \sum_{j \in Q_i} |x(\Theta(j))| \\ &= c \left(2\|k\|_\infty + 1\right)^{\gamma-n} \sum_{i=1}^{\|k\|_\infty} \sum_{m \in N_i} |x(m)| \\ &= c \left(2\|k\|_\infty + 1\right)^{\gamma-n(2\|k\|_\infty+1)^n} \sum_{i=1}^{(2\|k\|_\infty+1)^n} |x(i)|. \end{aligned}$$

Consequently,

$$M_\gamma(y)(m) \geq c \sup_{j \geq (2\|m\|_\infty+1)^n} j^{\frac{\gamma}{n}-1} \sum_{i=1}^j |x(i)|.$$

Hence, (12) followed on taking rearrangements. $\square$

Remark 1. For the classical fractional maximal operator, similar statements are proven in [3], and for the generalized fractional maximal operator, they are considered in [4].

From Theorem 2 and Theorem 3 follows the following.

Lemma 2. Let $n \in \mathbb{N}$, $\gamma \in [0, n)$ and $0 < q, p \leq \infty$. Assume that $\{v(m)\}$ and $\{w(m)\}$ are given nonnegative weight sequences. Then the operator M_γ is bounded from $\lambda_p(v)$ to $\lambda_q(w)$ ($M_\gamma : \lambda_p(v) \hookrightarrow \lambda_q(w)$) i.e.

$$\left(\sum_{k=1}^{\infty} ((M_\gamma y)^*(k))^q w(k) \right)^{\frac{1}{q}} \leq \tilde{C}_1 \left(\sum_{k=1}^{\infty} (y(k)^*)^p v(k) \right)^{\frac{1}{p}} \quad (13)$$

holds for any sequence $y = \{y(m)\}_{m \in \mathbb{Z}^n}$, if and only if the following inequality

$$\left(\sum_{m=1}^{\infty} \left(\sup_{m < k} k^{\frac{\gamma}{n}-1} \sum_{i=1}^k x(i) \right)^q w(m) \right)^{\frac{1}{q}} \leq \tilde{C}_2 \left(\sum_{k=1}^{\infty} x(k)^p v(k) \right)^{\frac{1}{p}} \quad (14)$$

holds for all nonnegative, nonincreasing sequences $\{x(n)\}_{n \in \mathbb{N}}$. Moreover, if $\tilde{C}_1$ and $\tilde{C}_2$ are the best constants in (13) and (14), respectively, then $\tilde{C}_1 \approx \tilde{C}_2$ and the multiplicative constants in all the equivalences above depend only on p, q .

Proof of Lemma 2. We will prove the equivalence of two statements:

The boundedness of the operator M_γ from $\lambda_p(v)$ to $\lambda_q(w)$, i.e., the fulfillment of inequality (13). The fulfillment of inequality (14) for all nonnegative, nonincreasing sequences $\{x(n)\}_{n \in \mathbb{N}}$.

(13) $\Rightarrow$ (14).

Assume that the operator M_γ is bounded, i.e. the inequality (13) holds for any sequence $\{y(m)\}_{m \in \mathbb{Z}^n}$. We will show that this implies inequality (14) for nonincreasing sequences $x(k)_{k \in \mathbb{N}}$. By Theorem 3 for given nonincreasing sequence $(x(k))_{k \in \mathbb{N}}$ there exists multivariable sequence $\{y(m)\}_{m \in \mathbb{Z}^n}$ such that $y^*(k) = x(k)$ $k \in \mathbb{N}$ and

$$(M_\gamma(y))^*(k) \geq D_2 \sup_{k \leq i < \infty} i^{\frac{\gamma}{n}-1} \sum_{j=1}^i x(j).$$

Therefore, from inequality (13) follows

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} \left(\sup_{k < i < \infty} i^{\frac{\gamma}{n}-1} \sum_{j=1}^i x(j) \right)^q w(k) \right)^{1/q} \\ & \leq D_2^{-1} \left(\sum_{k=1}^{\infty} (M_\gamma(y))^*(k)^q w(k) \right)^{\frac{1}{q}} \leq D_2^{-1} \tilde{C}_1 \left(\sum_{k=1}^{\infty} y^*(k)^p v(k) \right)^{\frac{1}{p}} \\ & = D_2^{-1} \tilde{C}_1 \left(\sum_{k=1}^{\infty} x(k)^p v(k) \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore, we obtain (14) with

$$\tilde{C}_2 \leq D_2^{-1} \tilde{C}_1.$$

(14) $\Rightarrow$ (13).

Assume that inequality (14) holds for all nonnegative, nonincreasing sequences $\{x(n)\}$. We will show that this implies the boundedness of the operator M_γ , i.e., inequality (13) for any sequence y .

Using Theorem 2 and (14) for nonincreasing sequence $((y^*)(k))_{k \in \mathbb{N}}$, we obtain

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} ((M_{\gamma}y)^*(k))^q w(k) \right)^{\frac{1}{q}} \\ & \leq D_1 \left(\sum_{k=1}^{\infty} \left(\sup_{k < i < \infty} i^{\frac{\gamma}{n}-1} \sum_{j=1}^i y^*(j) \right)^q w(k) \right)^{\frac{1}{q}} \\ & \leq D_1 \tilde{C}_2 \left(\sum_{k=1}^{\infty} (y^*(k))^p v(k) \right)^{\frac{1}{p}}. \end{aligned}$$

Therefore, we obtain (13) with

$$\tilde{C}_1 \leq D_1 \tilde{C}_2.$$

□

In the theorem below, we give a characterization for computing inequality (13) under various choices of the parameters p and q .

Theorem 4. Let $n \in \mathbb{N}$. $\gamma \in [0, n)$, and $0 < p, q < \infty$. Assume that $\{w(m)\}$ and $\{v(m)\}$ are given nonnegative real numbers not identically equal to zero. Then the inequality

$$\left(\sum_{k=1}^{\infty} ((M_{\gamma}x)^*(k))^q w(k) \right)^{\frac{1}{q}} \leq \tilde{C}_1 \left(\sum_{k=1}^{\infty} (x(k)^*)^p v(k) \right)^{\frac{1}{p}}$$

holds for any sequences $\{x(m)\}$ if and only if:

(i) If $1 < p \leq q$, then $\max\{C_0, C_1, C_2, C_3\} < \infty$, where

$$\begin{aligned} C_0 &= \left(\sum_{i=1}^{\infty} i^{\frac{\gamma q}{n}} w(i) \right)^{\frac{1}{q}} \left(\sum_{i=1}^{\infty} v(i) \right)^{-\frac{1}{p}}, \\ C_1 &= \sup_{k \in \mathbb{N}} k^{\frac{\gamma}{n}-1} W(k)^{\frac{1}{q}} \left(\sum_{m=1}^k \frac{V(m)^{\frac{1}{1-p}} v(m+1)}{m^{\frac{p}{1-p}} V(m+1)} \right)^{\frac{p-1}{p}}, \\ C_2 &= \sup_{k \in \mathbb{N}} \left(\sum_{i=k}^{\infty} i^{q(\frac{\gamma}{n}-1)} w(i) \right)^{\frac{1}{q}} \left(\sum_{m=1}^k \frac{V(m)^{\frac{1}{1-p}} v(m+1)}{m^{\frac{p}{1-p}} V(m+1)} \right)^{\frac{p-1}{p}}, \\ C_3 &= \sup_{k \in \mathbb{N}} \left(\sum_{i=1}^k w(i) i^{\frac{q\gamma}{n}} \right)^{\frac{1}{q}} V(k)^{-\frac{1}{p}}. \end{aligned}$$

(ii) If $p > 1$ and $q < p$, then $\max\{C_0, C_4, C_5, C_6, C_7\} < \infty$, where

$$\begin{aligned} C_4 &= \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} i^{q(\frac{\gamma}{n}-1)} w(i) \right)^{\frac{q}{p-q}} k^{q(\frac{\gamma}{n}-1)} w(k) \left(\sum_{m=1}^k \frac{V(m)^{\frac{1}{1-p}} v(m+1)}{m^{\frac{p}{1-p}} V(m+1)} \right) \right)^{\frac{p-1}{p}}, \\ C_5 &= \left(\sum_{k=1}^{\infty} W(k)^{\frac{q}{p-q}} w(k) \sup_{k \leq i < \infty} i^{(\frac{\gamma}{n}-1) \frac{pq}{p-q}} \left(\sum_{m=1}^k \frac{V(m)^{\frac{1}{1-p}} v(m+1)}{m^{\frac{p}{1-p}} V(m+1)} \right)^{\frac{(p-1)q}{p-q}} \right)^{\frac{p-q}{pq}}, \\ C_6 &= \left(\sum_{k=1}^{\infty} W(k)^{\frac{q}{p-q}} w(k) \sup_{k \leq i < \infty} i^{\frac{\gamma}{n} \frac{pq}{p-q}} V(i)^{-\frac{q}{p-q}} \right)^{\frac{p-q}{pq}}, \end{aligned}$$

$$\times \left(\sum_{m \geq i} V(m+1)^{-1} V(m)^{\frac{1}{1-p}} v(m+1) \right)^{\frac{(p-1)q}{p-q}} \frac{p-q}{pq}$$

$$C_7 = \left(\sum_{k=1}^{\infty} \left(\sum_{i=1}^k w(i) i^{\frac{q\gamma}{n}} \right)^{\frac{q}{p-q}} w(k) \sup_{k \leq i < \infty} i^{\frac{\gamma q}{n}} V(i)^{-\frac{q}{p-q}} \right)^{\frac{p-q}{pq}}.$$

(iii) If $0 < p \leq 1$ and $p \leq q$, then $\max\{C_8, C_9\} < \infty$, where

$$C_8 = \sup_{k \in \mathbb{N}} k^{\frac{\gamma}{n}-1} W(k)^{\frac{1}{q}} \sup_{1 \leq i \leq k} i V(i)^{-\frac{1}{p}}.$$

$$C_9 = \sup_{k \in \mathbb{N}} \left(\sum_{i=k}^{\infty} i^{q(\frac{\gamma}{n}-1)} w(i) \right)^{\frac{1}{q}} \sup_{1 \leq i \leq k} i V(i)^{-\frac{1}{p}}.$$

(iv) If $0 < q < p \leq 1$, then $\max\{C_{10}, C_{11}\} < \infty$, where

$$C_{10} = \left(\sum_{k=1}^{\infty} \left(\sum_{i=k}^{\infty} w(i) i^{q(\frac{\gamma}{n}-1)} \right)^{\frac{q}{p-q}} k^{q(\frac{\gamma}{n}-1)} w(k) \sup_{1 \leq i \leq k} i^{\frac{pq}{p-q}} V(i)^{\frac{q}{q-p}} \right)^{\frac{p-q}{pq}}$$

$$C_{11} = \left(\sum_{k=1}^{\infty} W(k)^{\frac{q}{p-q}} w(k) \sup_{k \leq i < \infty} i^{\frac{\gamma}{n} \frac{pq}{p-q}} V(i)^{\frac{q}{q-p}} \right)^{\frac{p-q}{pq}}.$$

Moreover, if we denote by $\tilde{C}$ the best constant in (14), then

$$\tilde{C} \approx \begin{cases} \max\{C_0, C_1, C_2, C_3\} & \text{in the case (i),} \\ \max\{C_0, C_4, C_5, C_6, C_7\} & \text{in the case (ii),} \\ \max\{C_8, C_9\} & \text{in the case (iii),} \\ \max\{C_{10}, C_{11}\} & \text{in the case (iv).} \end{cases}$$

Moreover, the multiplicative constants in all the equivalences above depend only on p, q .

Proof of Theorem 4. The proof easily follows from Lemma 2 by using [5, Theorem 2.3], with $u(i) = i^{\frac{\gamma}{n}-1}$, $a(m) = w(m)$ and $b(m) = v(m)$. It is easily seen that $z(k) = k^{\frac{\gamma}{n}-1}$ and $\sup_{i \leq j \leq n} u(j)j = n^{\frac{\gamma}{n}}$. Therefore, in the cases (iii) and (iv), we have one condition less than in [5, Theorem 2.3] because in case (iii) we have $C \approx C_8 + C_9 + C_{10}$ and $C_{10} \leq C_8$. In case (iv) we have $C \approx C_{11} + C_{12} + C_{13}$, $C_{13} \leq C_{12}$, therefor $C \approx C_{11} + C_{12}$, C_{11} and C_{12} are now C_{10} and C_{11} .

□

We note that weighted inequalities for Hardy operators and for other operators have also been considered in the literature [6]-[13].

4. Conclusion

In this paper, we have obtained sharp upper and lower bounds for the nonincreasing rearrangement of the discrete fractional maximal operator in weighted Lorentz-type sequence spaces $\lambda_p(v) \hookrightarrow \lambda_q(w)$. These estimates were derived using supramal operator techniques and reduction arguments applied to cones of monotone sequences.

In the paper [7]-[8], a continuous case is considered. In the discrete case, the result was problematic, and the result is essentially different. We have proved the new Copson theorem for the discrete case; in the results, it can be seen that it uses shifts - this is a new effect. Also, obtaining the estimate below for the decreasing rearrangement of the fractional maximal function is more difficult than in the continuous case. In the continuous case, it is enough to consider a radial function, there is no analog of a radial sequence in $\mathbb{Z}^n$.

Our results generalize and refine known weighted inequalities for discrete Hardy-type and kernel operators. The established equivalences involving iterated supremal expressions highlight a deep connection between discrete harmonic analysis and classical real-variable techniques.

From a practical perspective, such rearrangement-based estimates can be useful in problems related to smoothing analysis, approximation, and stability assessment in discrete computational settings.

Future research may focus on extending these results to generalized discrete potential-type operators and exploring their role in grid-based models.

5. Acknowledgements

The authors thank the reviewers for their careful reading of the manuscript and valuable comments.

Conflict of interest: the authors declare that there is no conflict of interest.

Authors' Contributions: the authors' contribution is equal.

References

- 1 C. Bennett and R. Sharpley, *Interpolation of Operators*. Pure and Applied Mathematics 129, Academic Press, Boston, MA, 1988, 469 p.
- 2 E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, 1970, 289 p. ISBN: 0-691-08079-8.
- 3 A. Cianchi, R. Kerman, B. Opic and L. Pick, A sharp rearrangement inequality for the fractional maximal operator. *Studia Mathematica*, 138 (2000), no. 3, 277–284.
- 4 N.A. Bokayev, A. Gogatishvili and A.N. Abek, On estimates of non-increasing rearrangement of generalized fractional maximal function. *Eurasian Mathematical Journal*, 14 (2023), no. 2, 13–23.
- 5 T. Ünver, A. Gogatishvili, N.A. Bokayev and N. Kuzebayeva, Weighted Hardy inequalities involving supremum for decreasing sequences. *Zeitschrift für Analysis und ihre Anwendungen*, 2026. <https://doi.org/10.4171/ZAA/1830>
- 6 A. Gogatishvili, M. Křepela, R. Ořhava and L. Pick, Weighted inequalities for discrete iterated Hardy operators. *Mediterranean Journal of Mathematics*, 17 (2020), no. 4, 132 p.
- 7 A. Gogatishvili, L. Pick and T. Ünver, Weighted inequalities for discrete iterated kernel operators. *Mathematische Nachrichten*, 295 (2022), no. 11, 2171–2196.
- 8 A. Gogatishvili and V.D. Stepanov, Reduction theorems for operators on the cones of monotone functions. *Journal of Mathematical Analysis and Applications*, 405 (2013), no. 1, 156–172.
- 9 A. Gogatishvili and V.D. Stepanov, Reduction theorems for weighted integral inequalities on the cone of monotone functions. *Uspekhi Mat. Nauk*, 2013, Vol. 68, no. 4 (412), 3–68; *Russian Math. Surveys*, 68 (2013), no. 4, 597–664.
- 10 K.G. Grosse-Erdmann, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*. Lecture Notes in Mathematics, Vol. 1679, Springer-Verlag, Berlin, 1998.
- 11 Р. Ойнаров и С.Х. Шалгинбаева, Весовые неравенства Харди на конусе монотонных последовательностей. *Известия Министерства науки Академии наук Республики Казахстан. Серия физико-математическая*, 1998, no. 1, 33–42.
- 12 E. Sawyer, Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96 (1990), no. 2, 145–158.
- 13 G. Sinnamon, Hardy's inequality and monotonicity, in *Function Spaces and Nonlinear Analysis*. Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague, 2005, 292–310 p.

Салмақталған Лоренц тізбектері кеңістіктері арасындағы дискретті бөлшекті максималды оператордың шенелгендігін сипаттау

А. Гогатишвили¹, Н.А. Бокаев², Н.К. Кузеубаева^{2*}

¹ Чехия ғылым академиясының математика институты, 115 67 Прага 1, Житна көшесі, 25, Чехия

² Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Сәтпаев көшесі, 2, 010008, Астана, Қазақстан

Аннотация. Бұл мақалада біз $\mathbb{Z}^n$, $n \in \mathbb{N}$, $\gamma \in [0, n)$, жиынында анықталған M_γ дискретті бөлшектік максималды операторының классикалық Лоренц тізбектер кеңістіктері $\lambda_p(v)$ мен $\lambda_q(w)$ арасында шенелген болуының толық сипаттамасын береміз, мұндағы $0 < p, q < \infty$, ал $\{v(m)\}$ және $\{w(m)\}$ — $\mathbb{N}$ жиынында анықталған теріс емес тізбектер. Алдымен біз $M_\gamma x$ дискретті бөлшектік максималды операторының өспейтін алмастыруы үшін дәл жоғарғы бағалауды аламыз, бұл бағалау дискретті Харди операторы мен $\{x(k)\}$ тізбегінің өспейтін алмастыруын қамтитын супремумдық оператор арқылы

өрнектеледі. Сондай-ақ, арнайы тізбектер үшін төменгі бағалауды келтіру арқылы алынған бағалаудың дәлдігін көрсетеміз. Осы нәтижені пайдалана отырып, салмақты Лоренц тізбектер кеңістіктері арасындағы M_γ дискретті бөлшектік максималды операторының шенелгендігін сипаттау мәселесін салмақты Лебег тізбектер кеңістіктері арасындағы дискретті супремум операторының шенелгендігін зерттеуге келтіруге болады. Аталған мәселе авторлардың жұмыстарында зерттелген, соның нәтижесінде дискретті бөлшектік максималды M_γ операторының дискретті салмақты Лоренц кеңістіктерінде шенелгендігіне түпкілікті сипаттама алуға мүмкіндік берді.

Түйін сөздер: бөлшекті максималды оператор, салмақталған Лоренц тізбектер кеңістігі, супремум операторы, Харди операторы.

Характеристика ограниченности дискретного дробно-максимального оператора между весовыми пространствами последовательностей Лоренца

А. Гогатишвили¹, Н.А. Бокаев², Н.К. Кузеубаева^{2*}

¹ Институт математики Чешской академии наук, ул.Житна, 25, 115 67 Прага 1, Чешская Республика

² Евразийский национальный университет им. Л.Н. Гумилёва, ул.Сатпаева, 2, 010008, Астана, Казахстан

Аннотация. В данной работе приводится полная характеристика ограниченности дискретного дробного максимального оператора M_γ , определенного на $\mathbb{Z}^n$, $n \in \mathbb{N}$, $\gamma \in [0, n)$, как оператора, действующего между классическими пространствами последовательностей Лоренца $\lambda_p(v)$ и $\lambda_q(w)$, где $0 < p, q < \infty$, а $\{v(m)\}$ и $\{w(m)\}$ неотрицательные последовательности, заданные на $\mathbb{N}$. Сначала мы получаем точную оценку сверху для невозрастающей перестановки последовательности $M_\gamma x$ (результата действия дискретного дробного максимального оператора) через супремальный оператор, включающий дискретный оператор Харди и невозрастающую перестановку последовательности $\{x(k)\}$, точность этой оценки подтверждается соответствующей оценкой снизу для специальных последовательностей. Используя этот результат, задачу о характеристике ограниченности дискретного дробного максимального оператора M_γ между дискретными весовыми пространствами последовательностей Лоренца можно свести к задаче об ограниченности дискретного супремального оператора между весовыми пространствами последовательностей Лебега, рассматриваемого на классе невозрастающих последовательностей. Эта задача была исследована в работах авторов, что позволило получить окончательную характеристику ограниченности дискретного дробного максимального оператора M_γ в дискретных весовых пространствах Лоренца.

Ключевые слова: дискретный дробно максимальный оператор, весовое пространство последовательностей Лоренца, супремальный оператор, оператор Харди.

References

- 1 C. Bennett and R. Sharpley, *Interpolation of Operators*. Pure and Applied Mathematics 129, Academic Press, Boston, MA, 1988, 469 p.
- 2 E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, New Jersey, 1970, 289 p. ISBN: 0-691-08079-8.
- 3 A. Cianchi, R. Kerman, B. Opic and L. Pick, A sharp rearrangement inequality for the fractional maximal operator. *Studia Mathematica*, 138 (2000), no. 3, 277–284.
- 4 N.A. Bokayev, A. Gogatishvili and A.N. Abek, On estimates of non-increasing rearrangement of generalized fractional maximal function. *Eurasian Mathematical Journal*, 14 (2023), no. 2, 13–23.
- 5 T. Ünver, A. Gogatishvili, N.A. Bokayev and N. Kuzeubayeva, Weighted Hardy inequalities involving supremum for decreasing sequences. *Zeitschrift für Analysis und ihre Anwendungen*, 2026. <https://doi.org/10.4171/ZAA/1830>
- 6 A. Gogatishvili, M. Křepela, R. Oľhava and L. Pick, Weighted inequalities for discrete iterated Hardy operators. *Mediterranean Journal of Mathematics*, 17 (2020), no. 4, 132 p.
- 7 A. Gogatishvili, L. Pick and T. Ünver, Weighted inequalities for discrete iterated kernel operators. *Mathematische Nachrichten*, 295 (2022), no. 11, 2171–2196.

- 8 A. Gogatishvili and V.D. Stepanov, Reduction theorems for operators on the cones of monotone functions. *Journal of Mathematical Analysis and Applications*, 405 (2013), no. 1, 156–172.
- 9 A. Gogatishvili and V.D. Stepanov, Reduction theorems for weighted integral inequalities on the cone of monotone functions. *Uspekhi Mat. Nauk*, 68 (2013), no. 4 (412), 3–68; *Russian Math. Surveys*, 68 (2013), no. 4, 597–664.
- 10 K.G. Grosse-Erdmann, *The Blocking Technique, Weighted Mean Operators and Hardy's Inequality*. Lecture Notes in Mathematics, Vol. 1679, Springer-Verlag, Berlin, 1998.
- 11 R.Oinarov and S.Kh. Shalginbaeva, Vesovye neravenstva Hardi na konuse monotonnykh posledovatel'nostei. [Weighted Hardy inequalities on the cone of monotone sequences]. *Izvestiya Ministerstva nauki Akademii nauk Respubliki Kazakhstan. Seriya fiziko-matematicheskaya* [Izv. Minister. Nauki Akad. Nauk Resp. Kaz. Ser. Fiz.-Mat.], 1998, no. 1, 33–42. [in Russian]
- 12 E. Sawyer, Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96 (1990), no. 2, 145–158.
- 13 G. Sinnamon, Hardy's inequality and monotonicity, in *Function Spaces and Nonlinear Analysis*. Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague, 2005, 292–310 p.

Information about the authors:

Amiran Gogatishvili – PhD, Leading researcher at the Institute of Mathematics, Institute of Mathematics of the Czech Academy of Sciences, Žitná, 25, 115 67 Praha 1, Czech Republic.

Nurzhan Adilkhanovich Bokayev – Doctor of Physical and Mathematical Sciences, Professor, L.N. Gumilyov Eurasian National University, Satpaev Str., 2, 010008, Astana, Republic of Kazakhstan.

Nurgul Kasymkhanovna Kuzebayeva – corresponding author, PhD student, L.N. Gumilyov Eurasian National University, Satpaev Str., 2, 010008, Astana, Republic of Kazakhstan.

Авторлар туралы мәлімет:

Амиран Гогатишвили – PhD, Математика институтының жетекші ғылыми қызметкері, Чехия ғылым академиясының математика институты, 115 67 Прага 1, Житна көшесі, 25, Чехия.

Нуржан Адильханович Бокаев – ф.-м.ғ.д., профессор, Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Сәтпаев көшесі, 2, 010008, Астана, Қазақстан.

Нургуль Касымхановна Кузеубаева – байланыс үшін автор, PhD докторант, Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Сәтпаев көшесі, 2, 010008, Астана, Қазақстан.

Received: 27.06.2026. After revision: 05.09.2026.

Approved: 18.09.2026. Available online: 27.09.2026.



Copyright: © 2026 by the authors. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY NC) license (<https://creativecommons.org/licenses/by-nc/4.0/>).