Bulletin of L.N. Gumilyov Eurasian National University. Mathematics, computer science, mechanics series, 2026, Vol. 154, №1, P. 34-46.
http://bulmathmc.enu.kz, E-mail: vest_math@enu.kz

Article
IRSTI: 27.23; 24.01.45; 27.01.33

Mathematical Modeling and Effective Properties in Dispersed Composites [1]

V. Mityushev
Faculty of Computer Science and Mathematics, Cracow University of Technology, Warszawska str., 24, Krakow, Poland
wladimir.mitiuszew@pk.edu.pl

Abstract. Determining the effective properties of dispersed composites, such as conductivity and permittivity, remains a central challenge in homogenization theory. This paper presents a comprehensive revision of the Self-Consistent Method and its modern variants, demonstrating that many so-called "new models" are, in fact, merely first-order approximations in concentration that reproduce the classical results of Maxwell, Clausius, Mossotti, and others. We establish a rigorous framework for assessing accuracy with respect to both inclusion concentration and contrast parameters. A particular role of simulations addresses to standard packages, when the opacity of numerical calculations often obscures the underlying probability distributions of inclusions. We conclude that the concept of a mathematical model is frequently misapplied in the study of dispersed random composites, especially when the fundamental principles of homogenization and asymptotic analysis are overlooked. Finally, we demonstrate how symbolic computation can be effectively employed to derive analytically correct formulas with precisely quantified accuracy.

Special attention is given to the relationship between analytical approaches and numerical simulations in the evaluation of effective material parameters. The paper emphasizes that reliable predictions require consistency with the fundamental principles of homogenization theory and the correct treatment of stochastic microstructures. By comparing classical and modern approaches, we identify the limits of applicability of commonly used approximations and clarify their asymptotic accuracy. The study also highlights the importance of transparent computational procedures, allowing the influence of inclusion geometry, concentration, and spatial distribution to be properly interpreted. The proposed methodology provides a unified perspective on dispersed composites and offers practical guidance for researchers working with effective medium theories, computational homogenization, and multiphase materials.

Keywords: Homogenization theory, dispersed composites, self-consistent method, asymptotic analysis, symbolic computation.

DOI: https://doi.org/10.32523/2616-7182/bulmathenu.2026/1.3

2000 Mathematics Subject Classification: 74Q15

[1]Зерттеу Қазақстан Республикасы Ғылым және жоғары білім министрлігі Ғылым комитетімен қаржыландырылған (грант №AP22686434)

1. Introduction

The theory of homogenization is a mathematical theory devoted to PDE with rapidly oscillated coefficients. It is based on physical consideration of heterogeneous media and used to derive macroscopic, averaged equations and physical coefficients [2, 34, 17]. It allows engineers and physicists to replace a heterogeneous material with a homogenized material to predict its overall behavior. The microscopic structure of a material yields its macroscopic response which can be expressed by numerical data and analytical formulas.

Since the 19th century, the works of Maxwell, Clausius, Mossotti, and many others have served as the bedrock for various composites [30, Chapter 9]. J.C. Maxwell proposed Self Consistent Method (SCM) [21] in 1873. The Maxwell-Garnett[2] approximation was derived in 1904. It is suitable for the complex dielectric permittivity. The same formula for real conductivity (permittivity) was known as the Clausius-Mossotti approximation published in 1850. It coincides with Maxwell's approximation. Perhaps the universality of mathematical modeling was not clear in the XIX century. The historical notes [19] and references therein clarify the series of the above approximations.

The modern literature is saturated with a rich assortment of different approximations called mathematical models that often lack theoretical justification. A fundamental problem in contemporary research is the misuse of the term "model." In many cases, it is used as a "magic word" to allow empirical reasoning where rigorous exact science should prevail [30]. This has resulted in a proliferation of formulas that sometimes contradict one another, violate the Hashin-Shtrikman bounds, or ignore the fundamental symmetry of tensors.

The well-known Hill approach based on the Representative Volume Element (RVE) for elastic composites [15, 16] was revisited in [9]. It was rigorously shown that the RVE must constitute a fundamental domain of a plane or spatial torus. For example, a flat torus can be represented as a parallelogram in the complex plane with identified opposite edges [25]. Alternative choices of RVE shape may lead to inconsistencies with the fundamental principles of homogenization as clearly shown in [29].

This paper seeks to restore mathematical rigor by analyzing the asymptotic and numerical precision [38]. We argue that for a mathematically defined problem, there should be a unique mathematical expectation for effective constants. Discrepancies in results, particularly in computer simulations, are not evidence of "different valid models," but rather a reflection of differing numerical protocols [12] and oversimplified approaches.

In this paper, we discuss the principles of mathematical modeling using a physical and mechanical problem concerning the macroscopic properties of stochastic dispersed media as an illustrative example. All relevant aspects are considered, including mathematical theories, empirical formulations, the use of standard numerical packages, and artificial intelligence (AI).

2. Mathematical Modeling

The most popular analytical formulas for the effective conductivity (permittivity) of dispersed composites are the Maxwell [21], Maxwell-Garnett [22] and Bruggeman [5] approximations, Mori-Tanaka method for elastic composites [31]. The famous neutral Hashin-Shtrikman assemblage [14] for coated composites and its extensions [36, 6, 7, 8] as well as the Keller [18]- Matheron [20]- Dykhne [10] approach and its extensions [33, 3], deserve a special consideration not discussed here. We will discuss analytical formulas for two-phase composites based on self-consistent method (SCM).

A typical modern review of various analytical formulas for the effective properties of composites, including elastic media, begins with a discussion of these formulas and their numerous extensions. Lots of items are laid out on the counter and offered for usage. Some contradict others; some violate the Hashin-Shtrikman bounds; some disturb the symmetry of tensors. However, everything becomes allowed due to the democratic using the word "model". There is no theoretical justification when creating a new model, giving way to empirical reasons. Experiments might confirm the model for some data.

[2] James Clerk Maxwell and J.C. Maxwell Garnett are different persons.

We will not discuss a rich assortment of models and narrow our focus to the specific two-phase 2D macroscopically isotropic composites in order to derive precision of various modifications of SCM such as effective medium approximation, mean-field, Mori-Tanaka methods [31], etc. Some popular modifications of SCM are covered by the first-order approximation of Schwarz's method. This investigation explains plenty of illusory different formulas which are reduced to the same Maxwell-type estimation for dilute composites. Some higher-order extensions violate homogenization principles and lead to methodologically misleading results [28, 29].

SCM has multiple meanings in different fields of science and engineering. We pay particular attention to SCM in the theory of composites by Maxwell [21] developed by Hill [15, 16] and many others, and its relations to Schwarz's method. S.G. Mikhlin developed the alternating method of Schwarz in 1949 [23]. Mikhlin's work can be considered a mathematical embodiment of Schwarz's approach.

The main result of the present paper consists of applying Schwarz's method and functional equations to composites and deriving higher-order approximations for the effective constants. In particular, we demonstrate that Maxwell's SCM and its modern modifications coincide with the first-order approximation of Schwarz's method.

Investigation of SCMs for dispersed composites is presented in [32], where such materials were referred to as “grain composites.” Following these studies, many engineers applied SCMs to compute numerical estimates of effective material constants for various composite systems. Nevertheless, the fundamental theoretical issue concerning the limitations of SCMs remained unresolved. The justification of SCM applicability has traditionally relied on the assumption of a dilute concentration of inclusions. It was observed that extending SCMs beyond this regime, specifically to higher-order terms in the inclusion volume fraction f, can lead to predictions that are physically inconsistent.

In [28], analytical expressions for effective properties were derived as power series expansions in the inclusion concentration f and the contrast parameter ϱ. These series were truncated to obtain polynomials in $f^{1/2}$ and ϱ, with coefficients explicitly determined and symbolically dependent on the geometry and spatial distribution of inclusions. Furthermore, the accuracy of these approximations was rigorously established as $O(|\varrho|^m f^{p/2})$, where m denotes the number of Schwarz method iterations and p characterizes the order of the concentration approximation used at each step.

This situation highlights a fundamental issue: why do different formulas arise for the same composite system? The underlying reason is often a misuse of terminology, where the concept of a “model” is incorrectly equated with that of a “formula.” In exact sciences, it is essential to distinguish clearly between the two. A mathematically well-defined problem should yield a unique solution; obtaining different results does not imply the existence of multiple models. A correct research methodology requires that the model be fully specified a priori, while the derivation of formulas is performed a posteriori. This principle also applies to probabilistic analyses of random composites, where the statistical description of inclusion distributions must be explicitly defined within the model. Only then can the expected values of effective properties be rigorously determined for the specified class of random media.

In many studies, random composites are simulated using algorithms that incorporate stochastic variables, the distributions of which are often implicitly embedded in computational procedures. Moreover, the results of such simulations may depend on the chosen numerical protocols. These factors can lead to different estimates of effective properties, reflecting variations in the assumed spatial distributions of inclusions.

The exact coefficients for the effective constants were derived in [11, 28] in the series in ϱ and f. This allows us to compare formulas from the previous publications with the asymptotic estimations to make conclusions concerning the validity of the formulas called "models." Such a comparison of the Maxwell formula (Clausius-Mossotti approximation) for equal disks yields the asymptotic formula for macroscopically isotropic composites

$$\lambda_e = \frac{1+\varrho f}{1-\varrho f} + O(f^3), \tag{1}$$

where λ_e denotes the macroscopic permittivity. The exact precision up to $O(f^3)$ was established in [27]. Equation (1) was extended to other shapes of inclusions. Some authors add the precision term $O(f^3)$ to such an approximation, however, it must be $O(f^{5/2})$ as demonstrated in [28].

The Hashin-Shtrikman bounds for 2D two-phase macroscopically isotropic composites for real normalized permittivity of components λ_1 and $\lambda = 1$ when $\lambda_1 > 1$ can be written in the form

$$\lambda_{HS}^- \le \lambda_e \le \lambda_{HS}^+, \tag{2}$$

where

$$\lambda_{HS}^- = 1 + \frac{2f(\lambda_1 - 1)}{(1-f)(\lambda_1 - 1) + 2}, \quad \lambda_{HS}^+ = \lambda_1 + \frac{2(1-f)(1-\lambda_1)\lambda_1}{f(1-\lambda_1) + 2\lambda_1}. \tag{3}$$

It was proved in that an extension of Maxwell's approach to a finite collection of inclusions (cluster) may give, at most, the effective properties of dilute clusters. Any general formula for 2D two-phase macroscopically anisotropic dispersed composites which does not include the location of disks holds up to $O(f^2)$; for macroscopically isotropic composites up to $O(f^{\frac{5}{2}})$. This criterion of lower order formula could be extended to multi-phase composites except at the Hashin-Shtrikman type assemblages with neutral inclusions. For instance, a set of valuable formulas for multi-phase macroscopically anisotropic composites is valid up to $O(f^{\frac{5}{2}})$.

A significant portion of modern "model creation" involves adding complex analytical terms to a basic first-order formula. One can take the standard term (1) and append an arbitrary tale:

$$\frac{1+\varrho f}{1-\varrho f} = \frac{1+\varrho f + 2.176\varrho^{\frac{7}{3}} f^{\frac{19}{2}}}{1-\varrho f + 3.143\varrho^{\frac{17}{4}} f^{\frac{19}{2}}} + 1.823\varrho^{\frac{13}{4}} f^{\frac{13}{2}} \ln f + O(f^2). \tag{4}$$

The additional terms on the right-hand side are introduced in an ad hoc manner. Although the formula above is formally correct, it lacks practical significance. While the right-hand side may appear to define a "new model," it does not extend beyond a first-order approximation in f.

Both the differential scheme and the Mori–Tanaka method can be derived using similar arguments. Consider equation (1) with explicit dependence on the variable f and introduce an increment from f to $f + \Delta f$ which induces a corresponding change from λ_e to $\lambda_e + \Delta\lambda_e$. Within the framework of the differential scheme, one assumes infinitesimally small increments. This allows one to pass to a differential equation for λ_e, whose solution yields a "new model." However, the principal limitation of this approach lies in the assumption that Δf is infinitesimal that does not yield the increasing of accuracy in f.

In Mori-Tanaka method, such an increment Δf typically implemented numerically. Tacitly, this method assumes that the behavior captured by the first-order approximation in f reflects the trend exhibited by higher-order approximations, see detailed discussion in [29].

This demonstrates that modifications such as Mori-Tanaka methods, effective medium approximations (EMA), and differential schemes are often just illusory different formulas that reduce to the same Maxwell (Clausius-Mossotti) estimation (1) for dilute composites. Applying such an approximation to non-dilute composites often results in physically unacceptable predictions.

3. Generalized alternating method of Schwarz and functional equations

The generalized alternating method of Schwarz, or shortly Schwarz's method, can be implemented in the form of different iterative schemes known in the theory of composites as the contrast and cluster expansions. The method was developed for the Riemenn-Hilbert boundary value problem for a finitely connected domain in [4]. A short description with using of the Poincar'e theta series can be found in [24].

The contrast expansion for 2D permittivity problems leads to a power series in ϱ. The cluster expansion means a series in $f^{1/2}$. Frequently, the authors of SCMs do not care about the precision in ϱ and $f^{1/2}$, and derive, in the best case, the same formula asymptotically equivalent to the Maxwell-Garnett formula. The analysis of precision demonstrate a discrepancy between the rigorously derived formulas for the effective permittivity of disks/spheres and some formulas obtained by SCM. Below, we demonstrate and explain this precision effect on a simple example of 2D two-phase conductive composite with circular inclusions.

Let $z = x + iy$ denote a complex variable. Let $\omega_1 = 1$ and $\omega_2 = i$ be the fundamental pair of periods on the complex plane $\mathbb{C}$. The fundamental parallelogram Q is defined by the vertices $\pm\frac{1}{2}$ and $\pm\frac{i}{2}$.

The points $m_1 + m_2 i$ ($m_1, m_2 \in \mathbb{Z}$) generate a doubly periodic lattice and the unit square cell

$$Q = Q_{(0,0)} = \left\{ z = t_1 + it_2 \in \mathbb{C} : -\frac{1}{2} < t_1, t_2 < \frac{1}{2} \right\}.$$

Let the points a_k belong to Q. Consider non-overlapping inclusions $D_k = \{|z - a_k| < r\}$ of conductivity λ $(k = 1, 2, \ldots, N)$ and the conductivity of the host be normalized to the unit. The conductivity λ is dimensionless and can be considered as the ratios of the conductivities of the k th inclusion to the conductivity of host.

We study conductivity of the doubly periodic composite when the host $D + m_1 + im_2\omega_2$ and the inclusions $D_k + m_1 + im_2$ are occupied by conducting materials. The functions $u(\mathbf{x})$ and $u_k(\mathbf{x})$ are harmonic in D and D_k ($k = 1, 2, \ldots, N$) and continuously differentiable in closures of the considered domains. The perfect contact on the interface has the form

$$u = u_k, \quad \frac{\partial u}{\partial \mathbf{n}} = \lambda \frac{\partial u_k}{\partial \mathbf{n}} \quad \text{on } L_k, \quad k = 1, 2, \ldots, N, \tag{5}$$

where $\frac{\partial}{\partial \mathbf{n}}$ denotes the outward normal derivative to $L_k = \partial D_k$. Let the external field be defined by the conditions

$$u(z + 1) = u(z) + 1, \; u(z + i) = u(z). \tag{6}$$

Introduce the contrast parameter

$$\varrho = \frac{\lambda - 1}{\lambda + 1}. \tag{7}$$

Introduce the double periodic complex flux

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \psi(z), \quad z \in D \tag{8}$$

and

$$\frac{\partial u_k}{\partial x} - i\frac{\partial u_k}{\partial y} = \frac{\lambda + 1}{2}\psi_k(z), \quad z \in D_k \tag{9}$$

We have

$$\psi(z + 1) = \psi(z), \quad \psi(z + i) = \psi(z). \tag{10}$$

Following [28, 26] we have

$$\psi\,(t) = \psi_k\,(t) + \frac{\varrho r^2}{t^2}\,\overline{\psi_k\,(t)}, \; t \in L_k, \; k = 1, 2, \ldots, N, \tag{11}$$

where the functions $\psi(z)$ and $\psi_k(z)$ are analytic in D and D_k ($k = 1, 2, \ldots, N$) and continuous in the closures of considered domains.

The $\mathbb{R}$-linear problem (10) - (11) can be reduced to a system of integral equations on the torus Q corresponding to Schwarz's method. In the case of circular inclusion, these integral equations become the functional equations [11]. Introduce the operator in a Banach space associated to continuous functions

$$\left(W^{(m)}_{m_1,m_2}F\right)(z) = \left(\frac{r}{z - a_m - m_1 - im_2}\right)^2 \overline{F\left(a_m + \frac{r^2}{\overline{z - a_m} + m_1 - im_2}\right)}. \tag{12}$$

Here, a function $F(z)$ is analytic in all the disks D_k and continuous in their closures. The norm $\|F\| = \max_k \max_{|z - a_k| \leq r} |F(z)|$.

Introduce for $m = k$

$$(J_{kk}F)(z) = \sum_{(m_1,m_2) \in \mathbb{Z}^2 \setminus (0,0)} \left(W^{(k)}_{m_1,m_2}F\right)(z) \tag{13}$$

and for $m \neq k$

$$(J_{km}F)(z) = \sum_{m_1,m_2 \in \mathbb{Z}} \left(W^{(m)}_{m_1,m_2} F \right)(z). \tag{14}$$

The double series (12)–(14) are defined by the Eisenstein summation that correctly determines their convergence [11].

The system of functional equations corresponding to the $\mathbb{R}$-linear problem (10) - (11) can be written in the form

$$\psi_k(z) = \varrho \sum_{m=1}^{N} {\sum}^{*}_{m_1,m_2 \in \mathbb{Z}} \left(W^{(m)}_{m_1,m_2} \psi_m \right)(z) + 1, \ |z - a_k| \leq r \ (k = 1, 2, \ldots, N), \tag{15}$$

where m_1, m_2 run over integers $\mathbb{Z}$ excluding $m_1 = m_2 = 0$ for $m = k$. It was justified that the system of functional equations (15) can be solved by a method of successive approximations absolutely convergent for $|\varrho| < 1$. It was proved that the uniform convergence holds for $|\varrho| = 1$, i.e., in the considered Banach space. Attention, the absolute convergence can fail in this case.

Equations (15) can be easily solved by use of symbolic computations. The infinite double summations in m_1, m_2 can be performed by application of the Eisenstein functions $E_n(z)$. Equations (15) can be solved by the contrast expansions in ϱ and by the expansions in $f = N\pi r^2$. This expansion equivalent to the expansion in r^2 can be written in the form

$$\psi_m(z) = \sum_{n=0}^{\infty} \psi_m^{(n)}(z) r^{2n} \quad |z - a_k| \leq r \ (k = 1, 2, \ldots, N). \tag{16}$$

where

$$\begin{aligned}
&\psi_m^{(0)}(z) = 1, \quad \psi_m^{(1)}(z) = \rho \textstyle\sum_{k=1}^{N} E_2(z - a_k), \\
&\psi_m^{(2)}(z) = \varrho^2 \textstyle\sum_{k,k_1=1}^{N} \overline{E_2(a_k - a_{k_1})} E_2(z - a_k), \\
&\psi_m^{(3)}(z) = \varrho^3 \textstyle\sum_{k,k_1,k_2=1}^{N} E_2(a_k - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})} E_2(z - a_{k_2}) \\
&\qquad -2\varrho^2 \textstyle\sum_{k,k_1=1}^{N} \overline{E_3(a_k - a_{k_1})} E_3(z - a_k),
\end{aligned} \tag{17}$$

It is assumed for shortness that $E_n(z - a_k)$ is replaced with

$$\widetilde{E}_n(z - a_k) = E_n(z - a_k) - (z - a_k)^{-n} \text{ for } k = m \tag{18}$$

and $E_n(a_k - a_m) := S_n$ for $k = m$, where the lattice sums for the square array $S_2 = \pi$, $S_3 = 0$, are used.

Following [28] the normalized effective conductivity of macroscopically isotropic composites can be found by the formula

$$\lambda_e = 1 + 2\varrho f \frac{1}{N} \sum_{m=1}^{N} \psi(a_m). \tag{19}$$

Substitute (17) into (19) and apply the asymptotic expansion in f

$$\lambda_e = 1 + 2\varrho f \left(1 + \frac{\varrho f}{\pi} e_2 + \frac{\varrho^2 f^2}{\pi^2} e_{22} + \frac{\varrho^2 f^3}{\pi^3} (\varrho \, e_{222} - 2 e_{33}) \right) + O(f^5), \tag{20}$$

where the following structural sums (basic sums or e-sums) are introduced [11, Chapter 4]

$$\begin{aligned} e_2 &= \frac{1}{N^2}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N} E_2(a_{k_0}-a_{k_1}), \\ e_{22} &= \frac{1}{N^3}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} E_2(a_{k_0}-a_{k_1})\overline{E_2(a_{k_1}-a_{k_2})}, \\ e_{33} &= \frac{1}{N^4}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} E_3(a_{k_0}-a_{k_1})\overline{E_3(a_{k_1}-a_{k_2})}, \\ e_{222} &= \frac{1}{N^4}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\sum_{k_3=1}^{N} E_2(a_{k_0}-a_{k_1})\overline{E_2(a_{k_1}-a_{k_2})}E_2(a_{k_2}-a_{k_3}). \end{aligned} \tag{21}$$

The analytical expressions (17) and (19)-(21) can be continued practically without limitations [11]. The same concerns 2D elastic problems and 3D conductivity problems with perfectly conducting spherical inclusions. In particular, all the terms for a regular lattice are written explicitly. Hence, this yields exact formulas for a regular array of circular inclusions [1]. It should be noted that other frequently used empirical and approximate formulas diverge with the known exact formulas.

The structural sums for a macroscopically isotropic composites satisfy some relations. In particular, $e_2 = \pi$ for ideally isotropic composites. This formula extend the famous Rayleigh's formula $S_2 = \pi$ concerning lattice sums [35, 37]. The structural sum e_{22} varies with the location of inclusions. It can be proved that e_{22} is positive and $e_{22} \geq \pi^2$.

The formula (20) yields a proof of the the Maxwell (Clausius-Mossotti) approximation (1) and exclude any not justified empirical continuation similar to (1).

It is interesting to discuss the contrast approximation. The second order approximation in ϱ follows from the application of two iterations to equation (15). We have the same zero and first order approximations in (20), namely,

$$\lambda_e = 1 + 2\varrho f + 2\varrho^2 f^2 + O(f^3) = \frac{1+\varrho f}{1-\varrho f} + O(f^3), \tag{22}$$

where $O(f^3)$ can be replaced with $O(\varrho^3)$. However, the third order terms are different.

Introduce the approximations in ϱ

$$\psi_m(z) = \sum_{n=0}^{\infty} \psi_m^{(n)}(z;\varrho)\varrho^n \quad |z-a_k| \leq r \ (k=1,2,\ldots,N), \tag{23}$$

and

$$\begin{aligned} &\psi_m^{(0)}(z;\varrho) = 1, \quad \psi_m^{(1)}(z) = r^2 \textstyle\sum_{k=1}^{N} E_2(z-a_k), \\ &\psi_m^{(2)}(z;\varrho) = \textstyle\sum_{l,k=1}^{N} \sum_{m_1,m_2\in\mathbb{Z}}^{*} \left(W_{m_1,m_2}^{(l)} E_2(z'-a_k)\right)(z), \ |z-a_m| \leq r \ (m=1,2,\ldots,N). \end{aligned} \tag{24}$$

The second order approximation includes the term

$$\left(W_{m_1,m_2}^{(l)} E_2(z'-a_l)\right)(z) = \left(\frac{r}{z-a_l-m_1-im_2}\right)^2 \overline{E_2\left(a_l - a_k + \frac{r^2}{\overline{z-a_l}-m_1+im_2}\right)}. \tag{25}$$

It follows from equation (19) that

$$
\begin{aligned}
&\tfrac{1}{N} \textstyle\sum_{m=1}^{N} \psi_m^{(2)}(a_m;\varrho) = \sum_{l,k=1}^{N} \sum_{m_1,m_2\in\mathbb{Z}}^{*} \left(\tfrac{r}{a_l-a_m+m_1+im_2}\right)^2 \times \\
&\overline{E_2\left(a_l-a_k+\tfrac{r^2}{\overline{a_m-a_l}-m_1+im_2}\right)}.
\end{aligned}
\tag{26}
$$

Using an expansion in r^2 following [11], one finds that this term contains the contribution $\frac{2\varrho^3 f^3}{\pi^2} e_{22}$. In addition, it includes infinitely many higher-order terms in f, for example, the term $-\frac{4\varrho^3 f^3}{\pi^3} e_{33}$ which also appears in (20), whether in f or in ϱ is ultimately immaterial, as both approaches lead to the same resulting double series. However, the first method proves to be significantly more efficient for symbolic computations.

4. Conclusion

The analysis presented in this paper demonstrates that many widely used self-consistent and mean-field approaches do not constitute independent mathematical models, but rather represent specific realizations of low-order asymptotic approximations. In particular, the classical Maxwell (Clausius–Mossotti) formula emerges as the universal first-order term with respect to the concentration of inclusions, while numerous modern extensions fail to improve their accuracy since they neglect asymptotic analysis.

A rigorous examination based on Schwarz's method reveals that the precision of such formulas for two-phases composites must be assessed simultaneously in terms of the concentration parameter f, the contrast parameter ϱ and the structural sums that encode the geometric characteristics of the microstructure. Within this framework, it becomes evident that many modifications including differential schemes, effective medium approximations, and Mori-Tanaka-type constructions do not extend validity beyond the dilute regime and may produce physically inconsistent predictions when extrapolated. Furthermore, the discussion highlights the necessity of a clear distinction between a mathematically defined model and an empirical formula. For well-posed deterministic or probabilistic descriptions of composites, the effective properties are uniquely defined, and discrepancies observed in the literature are typically attributable to violation of asymptotic analysis.

In summary, the reliability of analytical expressions for effective properties must be grounded in homogenization theory and asymptotic justification rather than heuristic "model building." Symbolic and analytical methods provide a consistent pathway to derive formulas with explicitly controlled accuracy, thereby restoring mathematical rigor to the study of dispersed composites.

It is interesting to observe an analogy between reviews generated by AI and those produced by certain groups of researchers who are primarily oriented toward the popularity of applied methods. In both cases, emphasis is often placed on methods with a large number of citations, despite well-known limitations in their applicability. This resembles a transition from skill to will [13], which may lead to AI hallucinations as well as similar effects within some research communities. In such cases, the high citation counts of works such as [5] and [31] (exceeding 11,000 citations at present) become a dominant argument for their continued inclusion in subsequent studies, overshadowing critical assessment.

If a group of people vote that $2 \times 2 = 5$, then they have produced a democratically agreed, but not a true mathematical statement. Mathematics is a dictatorship, see Principle of model and solution [30, page 223]. Its results are not decided by opinion or majority. We conclude this paper with a rhetorical question: May consensus including AI prevail over logic?

References

1 I. Andrianov and V. Mityushev, Exact and "exact" formulae in the theory of composites, Modern Problems in Applied Analysis, Springer International Publishing, Cham, 2018, 15–34.

2 N.S. Bakhvalov and G. Panasenko, Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials. Springer Science and Business Media, 2012.

3 W. Bielski and R. Wojnar, Plane flow through the porous medium with chessboard-like distribution of permeability, 2nd Workshop on Porous Media, 2018, 7.
4 B. Bojarski and V. Mityushev, R-linear problem for multiply connected domains and alternating method of Schwarz, Journal of Mathematical Sciences 189 (2013), no 1, 68–77.
5 D.A.G. Bruggeman, Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen. I. Dielektrizitatskonstanten und Leitfahigkeiten der Mischkorper aus isotropen Substanzen, Annalen der Physik 416 (1935), no 7, 636–664.
6 A. Cherkaev, Optimal three-material wheel assemblage of conducting and elastic composites, International Journal of Engineering Science 59 (2012), 27–39.
7 A. Cherkaev and A.D. Pruss, Effective conductivity of spiral and other radial symmetric assemblages, Mechanics of Materials 65 (2013), 103–109. https://doi.org/10.1016/j.mechmat.2013.05.004
8 A. Cherkaev, V. Mityushev, N. Rylko, and P. Kurtyka, The generalized Hashin–Shtrikman approach to Al/nano-TiC composite, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 478:2263 (2022). https://doi.org/10.1098/rspa.2022.0164
9 R. Czapla, P. Drygas, S. Gluzman, T. Ligocki, and V. Mityushev, Computational Strategy for Analyzing Effective Properties of Random Composites. Part II. Elasticity, Materials 18 (2025), 5041.
10 A.M. Dykhne, Conductivity of a two-dimensional two-phase system, Soviet Physics JETP 32 (1971), no 1, 63–65.
11 S. Gluzman, V. Mityushev, and W. Nawalaniec, Computational Analysis of Structured Media. Academic Press, Elsevier, Amsterdam, 2017.
12 S. Gluzman, Z. Zhunussova, A. Sarvarov, and V. Mityushev, Protocol-Dependent Critical Exponents in Random Composites: Beyond Universality, Symmetry 18 (2026), no 4, 700.
13 Grzonka, D. D. Grzonka, The New Engineer: Psychology, Systems, Open Source. Talk at the WordCamp Europe, Krakow, Poland, 4–6 June, 2026.
14 Z. Hashin and S. Shtrikman, A variational approach to the theory of the effective magnetic permeability of multiphase materials, Journal of Applied Physics 33 (1962), 3125–3131. https://doi.org/10.1063/1.1728579
15 R. Hill, Elastic properties of reinforced solids: Some theoretical principles, Journal of the Mechanics and Physics of Solids 11 (1963), 357–372.
16 R. Hill, A self-consistent mechanics of composite materials, Journal of the Mechanics and Physics of Solids 13 (1965), 213–222.
17 V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals. Springer Science & Business Media, 2012.
18 J.B. Keller, A theorem on the conductivity of a composite medium, Journal of Mathematical Physics 5 (1964), no 4, 548–549.
19 R. Landauer, Electrical conductivity in inhomogeneous media, AIP Conference Proceedings 40 (1978), no 1, 2–45.
20 G. Matheron, Eléments pour une théorie des milieux poreux. Masson, Paris, 1967.
21 J.C. Maxwell, A Treatise on Electricity and Magnetism, Vol. 1. Clarendon Press, 1873.
22 J.C.M. Maxwell-Garnett, Colours in Metal Glasses and in Metallic Films, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 203 (1904), no 359–371, 385–420.
23 S.G. Mikhlin, Integral Equations: and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, Vol. 4. Elsevier, 2014.
24 V. Mityushev, Hilbert boundary value problem for multiply connected domains, Complex Variables 35 (1998), 283–295.
25 V. Mityushev, R-linear problem on torus and its application to composites, Complex Variables 50 (2005), no 7–10, 621–630.
26 V.V. Mityushev, R-linear and Riemann–Hilbert Problems for Multiply Connected Domains, Advances in Applied Analysis. Springer Basel, Basel, 2012, 147–176.
27 V. Mityushev and P. Dryga's, Effective Properties of Fibrous Composites and Cluster Convergence, SIAM Journal on Multiscale Modeling and Simulation 17 (2019), no 2, 696–715. https://doi.org/10.1137/18M1184278
28 V. Mityushev and N. Rylko, Effective properties of two-dimensional dispersed composites. Part I. Schwarz's alternating method, Computers and Mathematics with Applications 111 (2022), 50–60.
29 V. Mityushev, Effective properties of two-dimensional dispersed composites. Part II. Revision of self-consistent methods, Computers and Mathematics with Applications 121 (2022), 74–84.
30 V. Mityushev, R.A. Kycia, W. Nawalaniec, and N. Rylko, Introduction to Mathematical Modeling and Computer Simulations. Chapman and Hall/CRC, 2025.
31 T. Mori and K. Tanaka, Average stress in matrix and average elastic energy of materials with misfitting inclusions, Acta Metallurgica 21 (1973), no 5, 571–574.
32 A.N. Norris, A differential scheme for the effective moduli of composites, Mechanics of Materials 4 (1985), no 1, 1–16.
33 Y.V. Obnosov, Exact solution of a boundary-value problem for a rectangular checkerboard field, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 452 (1996), no 1954, 2423–2442.

34 A. Bensoussan, J.L. Lions, and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Vol. 374. American Mathematical Society, 2011.
35 L. Rayleigh, On the influence of obstacles arranged in rectangular order upon the properties of a medium, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 34 (1892), no 211, 481–502.
36 K. Schulgasser, Sphere assemblage model for polycrystals and symmetric materials, Journal of Applied Physics 54 (1983), 1380–1382. https://doi.org/10.1063/1.332161
37 S. Yakubovich, P. Drygas, and V. Mityushev, Closed-form evaluation of two-dimensional static lattice sums, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 472 (2016), no 2195.
38 A.Z. Zhubanysheva, N. Temirgaliev, and Z.N. Temirgalieva, Application of divisor theory to the construction of tables of optimal coefficients for quadrature formulas, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 49 (2009), no 1, 14–25.

Дисперсті композиттердің математикалық модельдеуі мен тиімді қасиеттері

В. Митюшев
Компьютерлік ғылымдар және математика факультеті, Краков технология университеті, Варшава көш, 24, Краков, Польша

Аннотация. Дисперсті композиттердің тиімді қасиеттерін, мысалы, электрөткізгіштігі мен диэлектрлік өтімділігін анықтау гомогенизация теориясындағы негізгі мәселелердің бірі болып қала береді. Бұл мақалада өзін-өзі келісімді әдіске (Self-Consistent Method) және оның заманауи нұсқаларына жан-жақты талдау жасалады. Көптеген «жаңа модельдер» шын мәнінде қосындылар концентрациясы бойынша бірінші ретті жуықтаулар ғана екені және олардың Максвелл, Клаузиус, Моссотти және басқа ғалымдардың классикалық нәтижелерін қайта өндіретіні көрсетіледі. Қосындылар концентрациясы мен фазалар қасиеттерінің контраст параметрлерін ескере отырып, дәлдікті бағалаудың қатаң әдістемесі ұсынылады. Стандартты бағдарламалық пакеттерді қолданатын сандық модельдеуге ерекше назар аударылады, өйткені есептеулердің жабық сипаты көбінесе қосындылардың ықтималдық үлестірімдерін жасырып қалады. Зерттеу нәтижелері дисперсті кездейсоқ композиттерді талдауда математикалық модель ұғымының жиі қате қолданылатынын көрсетеді, әсіресе гомогенизация мен асимптотикалық талдаудың іргелі қағидалары ескерілмеген жағдайда. Сонымен қатар, аналитикалық тұрғыдан дұрыс және дәлдігі нақты бағаланған формулаларды шығару үшін символдық есептеулерді тиімді пайдалануға болатыны дәлелденеді.

Материалдардың тиімді параметрлерін анықтауда аналитикалық тәсілдер мен сандық модельдеу арасындағы өзара байланысқа ерекше көңіл бөлінеді. Жұмыста сенімді болжамдар жасау үшін гомогенизация теориясының іргелі қағидаларын сақтау және композиттердің стохастикалық микроқұрылымын дұрыс есепке алу қажеттігі көрсетілген. Классикалық және заманауи тәсілдерді салыстыру арқылы кеңінен қолданылатын жуықтаулардың қолданылу шектері анықталып, олардың асимптотикалық дәлдігі нақтыланады. Сондай-ақ есептеу рәсімдерінің ашықтығының маңыздылығы көрсетіліп, қосындылар геометриясының, олардың концентрациясының және кеңістіктік таралуының материалдың тиімді қасиеттеріне әсерін дұрыс түсіндіруге мүмкіндік беретіні атап өтіледі. Ұсынылған әдістеме дисперсті композиттерге біртұтас көзқарас қалыптастырып, тиімді орта теориялары, есептеуіш гомогенизация және көпфазалы материалдар саласында жұмыс істейтін зерттеушілерге практикалық ұсынымдар береді.

Түйін сөздер: гомогенизация теориясы, дисперсті композиттер, өзін-өзі келісімді әдіс, асимптотикалық талдау, символдық есептеулер.

Математическое моделирование и эффективные свойства дисперсных композитов

В. Митюшев

Факультет компьютерных наук и математики, Краковский университет технологии, ул. Варшава, 24, Краков, Польша

Аннотация. Определение эффективных свойств дисперсных композитов, таких как проводимость и диэлектрическая проницаемость, остается одной из центральных задач теории гомогенизации. В данной работе представлен всесторонний пересмотр метода самосогласования (Self-Consistent Method) и его современных модификаций. Показано, что многие так называемые «новые модели» в действительности являются лишь приближениями первого порядка по концентрации включений, воспроизводящими классические результаты Максвелла, Клаузиуса, Моссотти и других исследователей. Разработан строгий подход к оценке точности с учетом как концентрации включений, так и параметров контраста свойств фаз. Особое внимание уделено компьютерному моделированию с использованием стандартных программных пакетов, поскольку непрозрачность численных вычислений нередко скрывает лежащие в их основе вероятностные распределения включений. Сделан вывод о том, что понятие математической модели часто используется некорректно при исследовании случайных дисперсных композитов, особенно когда игнорируются фундаментальные принципы гомогенизации и асимптотического анализа. Наконец, показано, как символьные вычисления могут эффективно применяться для вывода аналитически корректных формул с точно контролируемой точностью.

Особое внимание уделяется взаимосвязи между аналитическими подходами и численным моделированием при определении эффективных параметров материалов. В работе подчеркивается, что надежные прогнозы требуют согласованности с фундаментальными принципами теории гомогенизации и корректного учета стохастической микроструктуры композитов. Сравнение классических и современных подходов позволяет установить границы применимости широко используемых приближений и уточнить их асимптотическую точность. Также показана важность прозрачных вычислительных процедур, обеспечивающих корректную интерпретацию влияния геометрии включений, их концентрации и пространственного распределения на эффективные свойства материала. Предлагаемая методология формирует единый взгляд на дисперсные композиты и предоставляет практические рекомендации для исследователей, работающих в области теорий эффективной среды, вычислительной гомогенизации и многофазных материалов.

Ключевые слова: теория гомогенизации, дисперсные композиты, метод самосогласования, асимптотический анализ, символьные вычисления.

References

1 I. Andrianov and V. Mityushev, Exact and “exact” formulae in the theory of composites, Modern Problems in Applied Analysis, Springer International Publishing, Cham, 2018, 15–34.

2 N.S. Bakhvalov and G. Panasenko, Homogenisation: averaging processes in periodic media: mathematical problems in the mechanics of composite materials. Springer Science and Business Media, 2012.

3 W. Bielski and R. Wojnar, Plane flow through the porous medium with chessboard-like distribution of permeability, 2nd Workshop on Porous Media, 2018, 7.

4 B. Bojarski and V. Mityushev, R-linear problem for multiply connected domains and alternating method of Schwarz, Journal of Mathematical Sciences 189 (2013), no 1, 68–77.

5 D.A.G. Bruggeman, Berechnung verschiedener physikalischer Konstanten von heterogenen Substanzen. I. Dielektrizitatskonstanten und Leitfahigkeiten der Mischkorper aus isotropen Substanzen, Annalen der Physik 416 (1935), no 7, 636–664.

6 A. Cherkaev, Optimal three-material wheel assemblage of conducting and elastic composites, International Journal of Engineering Science 59 (2012), 27–39.

7 A. Cherkaev and A.D. Pruss, Effective conductivity of spiral and other radial symmetric assemblages, Mechanics of Materials 65 (2013), 103–109. https://doi.org/10.1016/j.mechmat.2013.05.004

8 A. Cherkaev, V. Mityushev, N. Rylko, and P. Kurtyka, The generalized Hashin–Shtrikman approach to Al/nano-TiC composite, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 478:2263 (2022). https://doi.org/10.1098/rspa.2022.0164
9 R. Czapla, P. Drygas, S. Gluzman, T. Ligocki, and V. Mityushev, Computational Strategy for Analyzing Effective Properties of Random Composites. Part II. Elasticity, Materials 18 (2025), 5041.
10 A.M. Dykhne, Conductivity of a two-dimensional two-phase system, Soviet Physics JETP 32 (1971), no 1, 63–65.
11 S. Gluzman, V. Mityushev, and W. Nawalaniec, Computational Analysis of Structured Media. Academic Press, Elsevier, Amsterdam, 2017.
12 S. Gluzman, Z. Zhunussova, A. Sarvarov, and V. Mityushev, Protocol-Dependent Critical Exponents in Random Composites: Beyond Universality, Symmetry 18 (2026), no 4, 700.
13 Grzonka, D. D. Grzonka, The New Engineer: Psychology, Systems, Open Source. Talk at the WordCamp Europe, Krakow, Poland, 4–6 June, 2026.
14 Z. Hashin and S. Shtrikman, A variational approach to the theory of the effective magnetic permeability of multiphase materials, Journal of Applied Physics 33 (1962), 3125–3131. https://doi.org/10.1063/1.1728579
15 R. Hill, Elastic properties of reinforced solids: Some theoretical principles, Journal of the Mechanics and Physics of Solids 11 (1963), 357–372.
16 R. Hill, A self-consistent mechanics of composite materials, Journal of the Mechanics and Physics of Solids 13 (1965), 213–222.
17 V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals. Springer Science & Business Media, 2012.
18 J.B. Keller, A theorem on the conductivity of a composite medium, Journal of Mathematical Physics 5 (1964), no 4, 548–549.
19 R. Landauer, Electrical conductivity in inhomogeneous media, AIP Conference Proceedings 40 (1978), no 1, 2–45.
20 G. Matheron, Eléments pour une théorie des milieux poreux. Masson, Paris, 1967.
21 J.C. Maxwell, A Treatise on Electricity and Magnetism, Vol. 1. Clarendon Press, 1873.
22 J.C.M. Maxwell-Garnett, Colours in Metal Glasses and in Metallic Films, Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences 203 (1904), no 359–371, 385–420.
23 S.G. Mikhlin, Integral Equations: and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, Vol. 4. Elsevier, 2014.
24 V. Mityushev, Hilbert boundary value problem for multiply connected domains, Complex Variables 35 (1998), 283–295.
25 V. Mityushev, R-linear problem on torus and its application to composites, Complex Variables 50 (2005), no 7–10, 621–630.
26 V.V. Mityushev, R-linear and Riemann–Hilbert Problems for Multiply Connected Domains, Advances in Applied Analysis. Springer Basel, Basel, 2012, 147–176.
27 V. Mityushev and P. Dryga's, Effective Properties of Fibrous Composites and Cluster Convergence, SIAM Journal on Multiscale Modeling and Simulation 17 (2019), no 2, 696–715. https://doi.org/10.1137/18M1184278
28 V. Mityushev and N. Rylko, Effective properties of two-dimensional dispersed composites. Part I. Schwarz's alternating method, Computers and Mathematics with Applications 111 (2022), 50–60.
29 V. Mityushev, Effective properties of two-dimensional dispersed composites. Part II. Revision of self-consistent methods, Computers and Mathematics with Applications 121 (2022), 74–84.
30 V. Mityushev, R.A. Kycia, W. Nawalaniec, and N. Rylko, Introduction to Mathematical Modeling and Computer Simulations. Chapman and Hall/CRC, 2025.
31 T. Mori and K. Tanaka, Average stress in matrix and average elastic energy of materials with misfitting inclusions, Acta Metallurgica 21 (1973), no 5, 571–574.
32 A.N. Norris, A differential scheme for the effective moduli of composites, Mechanics of Materials 4 (1985), no 1, 1–16.
33 Y.V. Obnosov, Exact solution of a boundary-value problem for a rectangular checkerboard field, Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 452 (1996), no 1954, 2423–2442.
34 A. Bensoussan, J.L. Lions, and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Vol. 374. American Mathematical Society, 2011.
35 L. Rayleigh, On the influence of obstacles arranged in rectangular order upon the properties of a medium, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 34 (1892), no 211, 481–502.
36 K. Schulgasser, Sphere assemblage model for polycrystals and symmetric materials, Journal of Applied Physics 54 (1983), 1380–1382. https://doi.org/10.1063/1.332161
37 S. Yakubovich, P. Drygas, and V. Mityushev, Closed-form evaluation of two-dimensional static lattice sums, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 472 (2016), no 2195.
38 A.Z. Zhubanysheva, N. Temirgaliev, and Z.N. Temirgalieva, Application of divisor theory to the construction of tables of optimal coefficients for quadrature formulas, Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki 49 (2009), no 1, 14–25.

Information about the author:
Митюшев Владимир – Хабилит. докторы, проф., Компьютерлік ғылымдар және математика факультеті, Краков технология университеті, Варшава көш, 24, Краков, Польша.

Mityushev Vladimir – Dr hab., Prof., Faculty of Computer Science and Mathematics, Cracow University of Technology, Warszawska str., 24, Krakow, Poland.

Received: 03.02.2026. Revised: 21.02.2026.
Approved: 31.03.2026. Available online: 17.06.2026.