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SOME REMARKS ON LINEAR CLOSED SUBSPACES IN BERGMAN SPACES

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Abstract. Problems related to finite-dimensionality, compact embeddings, and quantitative estimates of dimensions of embedded subspaces in Bergman spaces play an important role in approximation theory, Banach space geometry, the theory of holomorphic function spaces, and the analysis of infinite-dimensional dynamical systems. Classical results of Grothendieck and Subramanian established fundamental principles for embeddings of subspaces in L_p -spaces, while recent studies have extended these ideas to Bergman spaces of holomorphic functions on bounded complex domains. In this context, the investigation of quantitative dimension estimates for closed subspaces embedded into Bergman spaces with stronger integrability conditions becomes particularly relevant.

In this article the finite-dimensionality Grothendieck type problem for closed linear subspaces of a Bergman space $A_p(\Omega, d\lambda)$ of holomorphic and p - integrable with respect to the Lebesgue measure $d\lambda$ on a bounded complex domain $\Omega \subset \mathbb{C}^n$, embedded into a Bergman space $A_q(\Omega, d\lambda)$ for $q > p \geq 1$, is analyzed. It is shown that if a closed linear subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda), q > p \geq 1$, its dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ proves to satisfy the numerical inequality $\frac{\omega_N^{2(q-1)/q}}{N} \frac{|\Omega|^{\frac{2-q}{q}}}{k_N(\xi_0)^{2(q-1)/q}} \leq \tilde{K}_{p,q}^2$ for some bounded constant $\tilde{K}_{p,q} > 0$, where $\omega_N = |SU(N)|$ is the volume of the compact unitary group $SU(N), |\Omega|$ is the volume of the bounded domain $\Omega \subset \mathbb{C}^n$ and $k_N(\xi_0) > 0-$ is the corresponding homogeneity parameter at $\xi_0 = (1, 0, 0, ..., 0)_N \in \partial\mathbb{D}^N \subset \mathbb{C}^N$.

Keywords: linear closed subspaces, Bergman space, functional Banach space, finite dimensionality, Lebesgue measures, embedding.

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1. Introduction

It is a classical problem of the Banach space theory to search for estimations of the dimension of linear closed subspaces of the functional Banach space $L_p\ (M; d\mu),\ p > 1,$ (in part, in $L_p(0,1;d\lambda)$, which has many applications in operator and approximation theories [1]- [10], in dynamical systems theory [11]-[17] and other applied fields. Here one can recall the known Subramanian inclusion theorem [18] of $L_p(M, d\mu)$ into $L_q(M, d\mu)$ for $0 < p < q$ and the classical Grothendieck theorem [19] on the estimation of the dimension of a linear closed subspace $S_p^{(\infty)} \subset L_p(M, d\mu)$, embedded into the Banach space $L_\infty(M, d\mu)$ with respect to a probability measure μ on a space M, as well as its generalization on the case of a linear closed subspace $S_p^{(c)} \subset C(M;\mathbb{R})$ of continuous functions, identically imbedded into $L_2(M; d\mu)$.

Recently, in November 2025, the authors of the work [20] announced the Grothendieck type embedding theorem for the Bergman space $A_p(\Omega, d\lambda) = Hol(\Omega) \cap L_p(\Omega)$ of holomorphic functions on a bounded domain $\Omega \subset \mathbb{C}^n$ and p - integrable on Ω with respect to the Lebesgue measure. They stated the following result.

Теорема 1. *Let $q > p \geq 1$ and for any bounded domain $\Omega \subset \mathbb{C}^n$ a closed subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ of the Bergman space $A_p(\Omega, d\lambda)$ be identically embedded into a Bergman space $A_q(\Omega, d\lambda)$. Then the dimension $\dim S_p^{(q)} < \infty$.*

The authors of this theorem based their proof on the following well known classical theorems [21]-[22] from functional analysis.

Теорема 2. *If a bounded operator $J : X \to Y$ from a Banach space X to a Banach space Y is compact and its image $\operatorname{Im} J \subset Y$ is closed, then $\dim (JX) < \infty$.*

Теорема 3. *(F. Riesz) A linear subspace of a normed space X is locally compact if and only if it is finite dimensional.*

As the authors of the above theorem did not obtain the upper estimation of the dimension dim $S_p^{(q)} = N < \infty$ of the subspace $S_p^{(q)} \subset A_p(\Omega; d\lambda)$ of the Bergman space $A_p(\Omega, d\lambda)$, embedded into the Bergman space $A_p(\Omega; d\lambda)$, we proposed a completely different approach to studying the Grothendieck embedding problem and presented an upper estimation of the closed subspace dimension under regard. In particular, we have stated the following dimension estimation theorem.

Теорема 4. *Let a linear closed subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda), p \geq 1,$ be identically embedded into a Banach space $A_q(\Omega; d\lambda)$ for $q > p \geq 1$, where $d\lambda$ is the Lebesgue measure on a bounded complex domain $\Omega \subset \mathbb{C}^n$. Then the dimension $\dim S_p^{(q)} = N \in \mathbb{N}$ of a closed subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ proves to satisfy the inequality $\frac{\omega_N^{\frac{2(q-1)}{q}} |\Omega|^{\frac{2-q}{q}}}{N k_N(\xi_0)^{\frac{2(q-1)}{q}}} \leq \tilde{K}_{p,q}^2$ for some bounded constant $\tilde{K}_{p,q} > 0$, where $\omega_N = |SU(N)|$ is the volume of the compact unitary group $SU(N), |\Omega|$ is the volume of the bounded domain $\Omega \subset \mathbb{C}^n$ and $k_N(\xi_0) > 0-$ is the corresponding homogeneity parameter at $\xi_0 = (1, 0, 0, ..., 0)_N \in \partial\mathbb{D}^N \subset \mathbb{C}^N$.*

In the special case $q = \infty$, when a closed linear subspce $S_p^{(\infty)} \subset A_p(\Omega, d\lambda), p \geq 1,$ is identically embedded into the Bergman space $A_\infty(\Omega, d\lambda)$, there is obtained the classical Grothendieck type theorem.

Теорема 5. *Let a linear closed topological subspace $S_p^{(\infty)} \subset A_p(\Omega, d\lambda), p \geq 1,$ be identically embedded into the Bergman space $A_\infty(\Omega, d\lambda)$, where $d\lambda$ is the Lebesgue measure on a bounded domain $\Omega \subset \mathbb{C}^n$. Then the dimension of the closed subspace $S_p^{(\infty)} \subset A_p(\Omega, d\lambda)$ proves to satisfy the inequality $\dim S_p^{(\infty)} = N \leq \tilde{K}_{p,\infty}^2$ for some bounded constant $\tilde{K}_{p,\infty} > 0$.*

2. Embedding of closed subspaces into Bergman spaces

Below we consider a closed linear subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda)$), allowing the identical embedding into the Bergman space $A_q(\Omega, d\lambda)$), where $q > p \geq 1$. Then the following theorems holds.

Теорема 6. *The Bergman spaces* $A_p(\Omega, d\lambda), p \geq 1$, *with respect to the Lebesgue measure* $d\lambda$ *on a bounded complex domain* $\Omega \subset \mathbb{C}^n$ *are Banach spaces.*

This theorem is based on the local precompactness of a complex bounded domain $\Omega \subset \mathbb{C}^n$ and the following classical Bergman estimation theorem.

Теорема 7. *For any compact subset* $V \subset \Omega$ *and arbitrary function* $f \in A_p(\Omega, d\lambda), p \geq 1$, *there exists such a bounded constant* $C_q(V) > 0$ *that*

$$\sup_V |f| \ < C_q(V)||f||_p. \tag{1}$$

Proof. Sketch. As the norm $||f||_p$ of any function $f \in A_p(\Omega, d\lambda), p \geq 1$, is a subharmonic function, the estimation (1) easily follows from the compactness of the subset $V \subset \Omega$ and the fact that the distance $dist(V, \partial\Omega) = r_V > 0$. □

Now we can formulate our main dimension $\dim S_p^{(q)} = N$ estimation theorem.

Теорема 8. *Let a linear closed subspace* $S_p^{(q)} \subset A_p(\Omega, d\lambda), p \geq 1$, *be identically embedded into a Banach space* $A_q(\Omega; d\lambda)$ *for* $q > p \geq 1$, *where* $d\lambda$ *is the Lebesgue measure on a bounded complex domain* $\Omega \subset \mathbb{C}^n$. *Then the dimension* $\dim S_p^{(q)} = N \in \mathbb{N}$ *of a closed subspace* $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ *proves to satisfy the inequality* $\frac{\omega_N^{\frac{2(q-1)}{q}} |\Omega|^{\frac{2-q}{q}}}{N k_N(\xi_0)^{\frac{2(q-1)}{q}}} \leq \tilde{K}_{p,q}^2$ *for some bounded constant* $\tilde{K}_{p,q} > 0$, *where* $\omega_N = |SU(N)|$ *is the volume of the compact unitary group* $SU(N), |\Omega|$ *is the volume of the bounded domain* $\Omega \subset \mathbb{C}^n$ *and* $k_N(\xi_0) > 0-$ *is the corresponding homogeneity parameter at* $\xi_0 = (1, 0, 0, ..., 0)_N \in \partial\mathbb{D}^N \subset \mathbb{C}^N$.

Let us consider a closed linear subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ of the functional Bergman space $A_p(M, d\mu)$, $p \geq 1$, with respect to the Lebesgue measure on Ω, satisfying, in addition, the identical embedding constraint $S_p^{(q)} \subset (A_p(\Omega, d\lambda); ||\cdot||_p) \hookrightarrow (A_q(\Omega, d\lambda); ||\cdot||_q)$ for $q > p \geq 1$. In order to state Theorem 8 we need some two lemmas.

Лемма 1. *For any* $q > p \geq 1$, *there exists a bounded positive constant* $K_{p,q} > 1$, *such that*

$$||f||_q \leq K_{p,q} \ ||f||_p \tag{2}$$

for any $f \in S_p^{(q)} \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda))$.

Proof. As the linear subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda)$, imbedded $A_q(\Omega, d\lambda)), q > p \geq 1$, is closed in $A_p(\Omega, d\lambda)$, one can define the identity imbedding mapping

$$J_p^{(q)} : S_p^{(q)} \subset A_p(\Omega, d\lambda) \to A_q(\Omega, d\lambda)). \tag{3}$$

If a sequence $\{f_n : n \in \mathbb{N}\} \subset S_p^{(q)}$ converges in $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ to an element $f \to S_p^{(q)} \hookrightarrow A_p(\Omega, d\lambda)$ with respect to the norm on $A_p(\Omega, d\lambda)$ and simultaneously its image $\{J_p^{(q)} f_n : n \in \mathbb{N}\} \subset S_p^{(q)} \subset A_q(\Omega, d\lambda)$ converges to an element $g \in A_q(\Omega, d\lambda) \subset A_p(\Omega, d\lambda)$ with respect to the norm on $A_q(\Omega, d\lambda)$, one can identify these limiting functions $f \sim g$ almost everywhere. Really, since $(\Omega, d\lambda) \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda), q > p \geq 1$, from the estimations

$$\begin{aligned} || \ f - g||_p \leq ||f - f_n||_p + ||g - f_n||_p \leq \\ \leq ||f - f_n||_p + || \ (g - f_n) \ ||_p \leq \\ \leq ||f - f_n||_p + || \left(g - J_p^{(q)} f_n\right) ||_q \ \overset{n \to \infty}{\to} 0 \end{aligned} \tag{4}$$

one obtains that $f \sim g$ almost everywhere and the image $J_p^{(q)}\ (S_p^{(q)}) \subset A_q(\Omega, d\lambda)$ is closed in $A_q(\Omega, d\lambda), q > p \geq 1.$ The latter, owing to the Banach closed graph theorem (see [21]-[25], gives rise to the existence of such a positive constant $K_{p,q} < \infty$ that

$$|| f||_q \leq K_{p,q}\ ||f||_p \tag{5}$$

for arbitrary $f \in S_p^{(q)} \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda), q > p \geq 1.$ Remark also, that the following estimations

$$||f||_2 \leq || f||_q = ||J_p^{(q)} f||_q \leq K_{p,q}\ ||f||_p < \infty \tag{6}$$

hold for any $f \in S_p^{(q)} \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda), q > p \geq 2,$ easily following from the Young inequality. □

Taking into account Lemma 1, we can formulate the next lemma, which is in some sense the converse to the inequality (2).

Лемма 2. *There exists a bounded constant* $\tilde{K}_{p,q} = K_{p,q}^p C_q(V)^{\frac{p-1}{p}} > 0,$ *such that the following inequality*

$$|| f||_{q,V} \leq \tilde{K}_{p,q}\ \ ||f||_{2,V} \tag{7}$$

holds for any compact $V \subset \Omega$ *and* $f \in S_p^{(q)} \hookrightarrow A_q(\Omega, d\lambda),\ \ q\ > p \geq 1(\neq 2).$

Proof. If $1 < p \leq 2,$ from the Young inequality

$$||f||_p \leq\ ||f||_2 \tag{8}$$

for any $f \in S_p^{(q)} \subset A_p(\Omega, d\lambda) \hookrightarrow A_q(\Omega, d\lambda)$ one obtains inequality (7) for the bounded $K_{p,q} > 0.$ If $p > 2,$ then for any compact $V \subset \Omega$ one can make use of the following inequality:

$$||f||_{p,V} \leq C_q(V)^{\frac{p-1}{p}} ||f||_{q,V}^{\frac{p-1}{p}}\ \ ||f||_{2,V}^{\frac{1}{p}}, \tag{9}$$

which holds for arbitrary $f \in S_p^{(q)} \subset A_p(\Omega, d\lambda).$ Now having put, by definition, $q\ > p \geq 1,$ the inequality (9) jointly with that of (2) gives rise to the searched estimation (7), where the constant $\tilde{K}_{p,q} = K_{p,q}^p C_q(V)^{\frac{p-1}{p}} > 0$ is bounded, thus proving the lemma. □

Proof. (*Proof of Theorem 8).* Based on the lemmas above, one can proceed to proving Theorem 8. First we can observe that inequality (7) can be estimated, owing to the classical Young inequality, from the below as

$$|l_\varphi(f)|_V \leq \tilde{K}_{p,q}\ \ ||f||_{2,V} \tag{10}$$

by means of a bounded linear functional $l_\varphi|_V : (S_p^{(q)}; ||\cdot||_{p,V}) \ \to \mathbb{C}$ on the Banach subspace $(S_p^{(q)}|_V; ||\cdot||_{q,V}),$ where $l_\varphi(f)|_V = (\varphi|f)|_V := \int_V \bar{\varphi} f d\lambda$ for some $\varphi \in (S_p^{(q)}|_V; ||\cdot||_{q,V})' \simeq\ (S_p^{(q)}|_V; ||\cdot||_{\tilde{q},V}),\ \ 1/\tilde{q} + 1/q = 1,$ under the constraint $||\varphi||_{\tilde{q},V} = 1.$ Taking inequality (10) and the evident embedding condition $(S_p^{(q)}; ||\cdot||_{2,V}) \subset (S_p^{(q)}; ||\cdot||_{\tilde{q},V}),$ one can calculate that

$$\sup_{||f||_{2,V} \neq 0} \frac{|l_\varphi(f)|_V}{||f||_{2,V}} = ||\varphi||_{2,V} \leq \tilde{K}_{p,q}. \tag{11}$$

If now to choose an orthonormal basis $\Phi|_V = \{\varphi_1, \varphi_2, ..., \varphi_N\}|_V \subset (S_p^{(q)}|_V; ||\cdot||_{2,V})$ for some $N \in \mathbb{N},\ \ (\varphi_j|\varphi_k)|_V =\ \int_V\ \bar{\varphi}_j \varphi_k d\lambda =\ \delta_{jk}, ||\varphi_j||_{2,V} = 1, j,k = \overline{1,N},$ one can observe that a function $\varphi_a|_V := \langle a|\varphi\rangle_{N,V} =\ \sum_{j=1}^N \bar{a}_j \varphi_j|_V\ \ \in\ \ (S_p^{(q)}|_V; ||\cdot||_{2,V})$ has the norm

$$||\varphi_a||_{2,V} = \left(\sum_{j=1}^N |a|^2 \right)^{1/2} = |a|_N, \tag{12}$$

where the vector $\varphi|_V=(\varphi_1,\varphi_2,...,\varphi_N)^\intercal|_V\in\left(S_p^{(q)}|_V\right)^N$ and took $a\in\mathbb{E}^N$, as an arbitrary vector. Having substituted the value of the norm (12) into (11), one obtains the inequality

$$|a|_N\ \leq \tilde{K}_{p,q}, \tag{13}$$

which should be combined with the imposed above condition $||\varphi_a||_{\tilde{q},V}=1$. Taking into account that

$$||\varphi_a||_{\tilde{q},V}^{\tilde{q}}=\int_V|\langle a|\varphi\rangle_N|^{\tilde{q}}d\lambda=|a|_N^{\tilde{q}}\int_V|\langle\xi|\varphi\rangle_N|^{\tilde{q}}d\lambda=1, \tag{14}$$

where $a\in\mathbb{E}^N$ and $\xi:=a/|a|_N\ \in\partial\mathbb{D}^N,|\xi|_N=1$, we can get rid of the variables $\xi\in\ \partial\mathbb{D}^N$, if to apply to the above norm equality (14) the averaging method [26]-[27] over the unit sboundary $\partial\mathbb{D}^N$, which is equivalent to double averaging over the compact Lie group $SU(N)$, as all vectors $\xi\in\ \partial\mathbb{D}^N$ allow the representation $\xi=\xi_0 g\in\partial\mathbb{D}^N$ at the fixed wector $\xi_0=(1,0,0,...,0)_N$ $\in\partial\mathbb{D}^N\subset\mathbb{C}^N,|\xi_0|=1$, and $g\in SU(N)$. Namely, by integrating it with respect to the compact group measure $d\omega_N(g),g\in SU(N)$:

$$\begin{gathered}|a|_N^{\tilde{q}}\int_V d\lambda\left(\int_{SU(N)}d\omega_N(g)|\langle\xi_0 g|\varphi\rangle_N|^{\tilde{q}}\right)=|a|_N^{\tilde{q}}k_N(\xi_0)\int_V d\lambda|\varphi|_N^{\tilde{q}}\ =\\ =|a|_N^{\tilde{q}}\ ||\ |\varphi|_N||_{\tilde{q},V}^{\tilde{q}}\ \ k_N(\xi_0)\leq\ \omega_N,\end{gathered} \tag{15}$$

where $k_N(\xi_0)>0$ is the corresponding homogeneity parameter at $\xi_0=(1,0,0,...,0)_N\in\partial\mathbb{D}^N$ $\subset\mathbb{C}^N$, and $\omega_N=|SU(N)|=\int_{SU(N)}\ d\omega_N(g)$ denotes the compact $SU(N)$ - group volume, we can equivalently obtain from (15) that

$$|a|_N=\frac{\omega_N^{1/\tilde{q}}}{||\ |\varphi|_N||_{\tilde{q},V}k_N(\xi_0)^{1/\tilde{q}}}. \tag{16}$$

Since the norm $||\ \ |\varphi|_N||_{\tilde{q},V}\leq||\ \ |\varphi|_N||_{2,V}|\Omega|^{\frac{q-2}{2q}}=\left(\int_V\langle\bar{\varphi}|\varphi\rangle_N d\mu\right)^{1/2}|\Omega|^{\frac{q-2}{2q}}=N^{1/2}|\Omega|^{\frac{q-2}{2q}}$, where we have denoted the bounded domain volume $|\Omega|:=\int_\Omega d\lambda$, the equality (16) jointly with the condition (13) yields the final numerical estimation

$$\frac{\omega_N^{2(q-1)/q}}{N}\frac{|\Omega|^{\frac{2-q}{q}}}{k_N(\xi_0)^{2(q-1)/q}}\leq\tilde{K}_{p,q}^2, \tag{17}$$

whose left hand side is bounded for those integers $N\in\mathbb{N}$, which ensure the embedded subspace $S_p^{(q)}\subset A_p(\Omega,d\lambda)\hookrightarrow\ A_q(\Omega,d\lambda)$ at given $q\geq p$, to be finite dimensional, that is dim $S_p^{(q)}=N<\infty$, thus proving the theorem.

Regarding the critical case $q=\infty$, since the representation (10) is not more acceptable, the linear bounded functional used there should be replaced by the following natural expression:

$$|l_x(f)|_V\ \leq\tilde{K}_{p,\infty}||f||_{2,V} \tag{18}$$

for any $f\in S_p^{(\infty)}\subset A_\infty(\Omega,d\lambda)$ and arbitrary compact $V\subset\Omega$, where at $x\in V$ the value $l_x(f)|_V:=f(x)|_V\in\mathbb{C}$. Having calculated the value $\sup_{||f||_{2,V}\neq0}\frac{|l_x(f)|_V}{||f||_{2,V}}=||l_x||_V\ \leq\tilde{K}_{p,\infty}$ and using the Riesz representation theorem for the functional $l_x:(S_p^{(\infty)}|_V;||\cdot||_{2,V})\to\mathbb{C}$ on the Hilbert subspace $(S_p^{(\infty)}|_V;||\cdot||_{2,V})\subset(A_p(V,d\lambda);||\cdot||_{2,V})$, there exists for any $x\in V\subset\Omega$ such a function $g_x\in\ (S_p^{(\infty)}|_V;||\cdot||_{2,V})$ that

$$l_x(f)|_V=(g_x|f)|_V \tag{19}$$

and $||l_x||_V\ =||g_x||_{2,V}$ for all $f\in(S_p^{(\infty)}|_V;||\cdot||_{2,V})$. If now $\Phi_p^{(\infty)}|_V:=\{\varphi_1,\varphi_2,...,\varphi_N,...\}|_V$ $\subset(S_p^{(\infty)}|_V;||\cdot||_{2,V})$ is a complete orthonormal set of functions, that is $||\varphi_j||_{2,V}=1$, $(\varphi_j|\varphi_k)|_V=\int_V\ \bar{\varphi}_j\varphi_k d\mu=\ \delta_{jk},j,k\in\mathbb{N}$, the related Parceval equality

$$||g_x||_{2,V}^2=\sum_{j\in\mathbb{N}}|(g_x|\varphi_j)|_V^2=\sum_{j\in\mathbb{N}}|\varphi_j(x)|_V^2 \tag{20}$$

combined with the inequality (19) yields the next one:

$$\sum_{j\in\mathbb{N}} |\varphi_j(x)|^2 \le \tilde{K}^2_{p,\infty}, \tag{21}$$

which holds for any $x \in V \subset \Omega$. Having integrated the obtained inequality (21) over the whole space $V \subseteq \Omega$, we obtain that

$$\text{card } \Phi_p^{(\infty)} = N \le \tilde{K}^2_{p,\infty} \tag{22}$$

for some $N = \dim S_p^{(\infty)}$. □

The reasonings above, concerning the special case $q = \infty$, can be reformulated as the following theorem.

Теорема 9. *Let a linear closed subspace* $S_p^{(\infty)} \subset A_p(\Omega, d\lambda), p \ge 1,$ *be identically embedded into a Banach space* $A_\infty(\Omega, d\lambda)$, *where* $d\lambda$ *is the Lebesgue measure on a bounded complex domain* $\Omega \subset \mathbb{C}^n$. *Then the dimension of the closed subspace* $S_p^{(\infty)} \subset A_p(\Omega, d\lambda)$ *proves to satisfy the inequality* $\dim S_p^{(\infty)} = N \le \tilde{K}^2_{p,\infty}$ *for some bounded constant* $\tilde{K}_{p,\infty} > 0$.

3. Conclusion

We considered closed linear subspaces $S_p^{(q)} \subset A_p(\Omega, d\lambda)$ of the functional Bergman space $(A_p(\Omega, d\lambda); ||\cdot||_p)$, $p > 1$, allowing the identical embedding into the Bergman space $(A_q(\Omega, d\lambda)); ||\cdot||_q)$, $q > p \ge 1$, regarding the Lebesgue measure $d\lambda$ on a bounded complex domain $\Omega \subset \mathbb{C}^n$. Based on the averaging technique on the unitary Lie group $SU(N)$, we derived the numerical estimation $\frac{\omega_N^{2(q-1)/q}}{N} \frac{|\Omega|^{\frac{2-q}{q}}}{k_N(\xi_0)^{2(q-1)/q}} \le \tilde{K}^2_{p,q}$ for the dimension $\dim S_p^{(q)} = N$ of a closed linear subspace $S_p^{(q)} \subset A_p(\Omega, d\lambda), p \ge 1$, embedded into a Bergman space $A_q(\Omega, d\lambda), q > p \ge 1$, where $\tilde{K}_{p,q} > 0$ is some some bounded constant, $\omega_N = |SU(N)|$ is the volume of the compact unitary group $SU(N), |\Omega|$ is the volume of the bounded domain $\Omega \subset \mathbb{C}^n$ and $k_N(\xi_0) > 0-$ is the corresponding homogeneity parameter at $\xi_0 = (1, 0, 0, ..., 0)_N \in \partial\mathbb{D}^N \subset \mathbb{C}^N$.

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Бергман кеңістіктеріндегі тұйық сызықтық ішкі кеңістіктер туралы кейбір ескертулер

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Аннотация. Гротендик пен Субраманианның классикалық нәтижелері L_p кеңістіктеріне ішкі кеңістіктерді енгізудің іргелі қағидаларын қалыптастырды, ал кейінгі зерттеулер бұл идеяларды шектелген комплекс облыстарында анықталған голоморфты функциялардың Бергман кеңістіктеріне жалпылады. Осы тұрғыдан алғанда, интегралдану шарттары күштірек болатын Бергман кеңістіктеріне енгізілген тұйық ішкі кеңістіктердің өлшемділігіне қатысты сандық бағаларды зерттеу ерекше өзектілікке ие.

Бұл жұмыста шенелген $\Omega \subset \mathbb{C}^n$ облысында Лебег өлшеміне қатысты p-интегралданатын голоморфты функциялардың $A_p(\Omega, d\lambda)$ Бергман кеңістігінің тұйық сызықтық ішкі кеңістіктері үшін Гротендик типіндегі ақырлы өлшемділік мәселесі қарастырылады. Атап айтқанда, $q > p \geq 1$ жағдайында $A_p(\Omega, d\lambda)$ кеңістігінің $A_q(\Omega, d\lambda)$ Бергман кеңістігіне үзіліссіз енгізілетін тұйық сызықтық ішкі кеңістіктері зерттеледі. Кез келген $S_p^{(q)} \subset A_p(\Omega, d\lambda)$, $S_p^{(q)} \hookrightarrow A_q(\Omega, d\lambda)$ тұйық сызықтық ішкі кеңістігінің ақырлы өлшемді болатыны дәлелденді. Сонымен қатар, $\dim S_p^{(q)} = N$ болғанда, оның өлшемділігі үшін

$\frac{\omega_N^{2(q-1)/q}}{N}\frac{|\Omega|^{\frac{2-q}{q}}}{k_N(\xi 0)^{2(q-1)/q}} \leq \tilde{K}p, q^2$ сандық бағалауы алынады, мұндағы $\omega_N = |SU(N)|$ — $SU(N)$ ықшам арнайы унитарлық тобының көлемі, $|\Omega|$ — Ω облысының көлемі, $k_N(\xi_0)$ — $\xi_0 = (1, 0, \ldots, 0)N \in \partial\mathbb{D}^N$ нүктесіне сәйкес келетін біртектілік параметрі, ал $\widetilde{K}p, q > 0$ — енгізу тұрақтысы.

Түйін сөздер: тұйық сызықтық ішкі кеңістіктер, Бергман кеңістігі, Банах функциялық кеңістігі, ақырлы өлшемділік, Лебег өлшемі, енгізу.

Некоторые замечания о замкнутых линейных подпространствах в пространствах Бергмана

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Аннотация. Классические результаты Гротендика и Субраманиана заложили фундаментальные принципы вложения подпространств в пространства L_p, тогда как более поздние исследования распространили эти идеи на пространства Бергмана голоморфных функций, определённых на ограниченных комплексных областях. В этом контексте особую актуальность приобретает исследование количественных оценок размерности замкнутых подпространств, вложенных в пространства Бергмана с более сильными условиями интегрируемости.

В работе исследуется задача типа Гротендика о конечномерности для замкнутых линейных подпространств пространства Бергмана $A_p(\Omega, d\lambda)$ голоморфных функций, p-интегрируемых относительно меры Лебега на ограниченной области $\Omega \subset \mathbb{C}^n$, непрерывно вложенных в пространство Бергмана $A_q(\Omega, d\lambda)$ при $q > p \geq 1$. Доказано, что всякое замкнутое линейное подпространство $S_p^{(q)} \subset A_p(\Omega, d\lambda)$, $S_p^{(q)} \hookrightarrow A_q(\Omega, d\lambda)$, является конечномерным. Более того, если $\dim S_p^{(q)} = N$, то размерность N удовлетворяет количественной оценке $\frac{\omega_N^{2(q-1)/q}}{N}\frac{|\Omega|^{\frac{2-q}{q}}}{k_N(\xi 0)^{2(q-1)/q}} \leq \tilde{K}p, q^2$, где $\omega_N = |SU(N)|$ — объём компактной унитарной группы $SU(N)$, $|\Omega|$ — объём области Ω, $k_N(\xi_0)$ — соответствующий параметр однородности в точке $\xi_0 = (1, 0, \ldots, 0)_N \in \partial\mathbb{D}^N$, а $\widetilde{K}_{p,q} > 0$ — константа вложения.

Ключевые слова: замкнутые линейные подпространства, пространство Бергмана, банахово функциональное пространство, конечномерность, мера Лебега, вложение.

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