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ON WEAK CONVERGENCE OF EMPIRICAL MEASURES FOR SETS MINIMIZING THE HAUSDORFF DISTANCE

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Abstract. This paper investigates the asymptotic behavior of finite sets that minimize the Hausdorff distance to a compact metric space. The primary objective is to establish conditions under which the empirical measures supported on these optimal sets converge weakly to the normalized Hausdorff measure. To facilitate this, we introduce the class of uniformly asymptotically open convex Euclidean metrics. These metrics characterize spaces that exhibit a local Euclidean structure in an asymptotic sense, allowing for a rigorous analysis of the geometry of Voronoi cells. The paper provides a sufficient reformulation of weak convergence based on the concept of sets uniformly eating Voronoi boundaries with respect to a regular measure. A complete proof of weak convergence to the normalized one-dimensional Hausdorff measure is presented for the specific case of a connected one-dimensional manifold (the circle). While the one-dimensional case yields a clear result due to the simple structure of Voronoi boundaries, the paper concludes by noting that higher-dimensional cases remain an open challenge due to the increased geometric complexity of the resulting Voronoi cells.

Keywords: Hausdorff distance, empirical measures, quantization, Voronoi cells, weak convergence, metric compacta

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1. Introduction

Let $\langle S_\rho, \rho\rangle$ be an infinite compact metric space. A central problem in quantization theory and location analysis is the study of sets $B_k \subseteq S_\rho$ with $|B_k| \leq k$ that minimize the Hausdorff distance $\rho_H(B_k)$ among all sets of cardinalities at most k to the whole space S_ρ. It is intuitively clear that for a reasonable surface, the empirical measures supported on such optimal sets must converge weakly to the corresponding Hausdorff measures. In this note, we prove a result of this type for a class of topological manifolds with metrics that behave locally like Euclidean ones.

2. A Sufficient Reformulation of Weak Convergence

For the definitions and properties of weak convergence and Hausdorff measures, see [2].

Definition 1. A sequence of finite sets $B_k \subseteq S_\rho$ is called shrinking if $\lim_{k\to\infty} \rho_H(B_k) = 0$.

Definition 2. For a finite set $B \subseteq S_\rho$, a number $q \in \mathbb{R}^+$, and a point $x \in B$, the set of all points $p \in S_\rho$ whose distance to x is at most q times the distance to the set $B \setminus \{x\}$ is called the fluctuated Voronoi cell of the point x associated with the set B, the number q and the metric ρ, it is denoted by $V_\rho^q B(x)$.

Denote by $V_\rho^{q,0} B(x) = \bigcup_{s \in (0,q)} V_\rho^s B(x) \subseteq V_\rho^q B(x)$ the open fluctuated Voronoi cell. Let $W_\rho B(A)$ be the union of all Voronoi cells of B that have non-empty intersections with a set $A \subseteq S_\rho$; this union is called the Voronoi closure of A.

Definition 3. A sequence of finite sets $B_k \subseteq S_\rho$ is called uniformly eating Voronoi boundaries with respect to a Borel measure μ on S_ρ if

$$\lim_{k\to\infty} \sup_{b \in B_k} \left| \frac{\mu(V_\rho^1 B_k(b))}{\mu(V_\rho^{1,0} B_k(b))} - 1 \right| = 0.$$

Lemma 1. *For a sequence B_k that is uniformly eating Voronoi boundaries with respect to a measure μ and for any sequence of sets A_k, the following limits hold:*

$$\lim_{k\to\infty} \frac{\sum_{p \in B_k \cap W_\rho B_k(A_k)} \mu(V_\rho^{1,0} B_k(p))}{\mu(W_\rho B_k(A_k))} = 1 \quad \textit{and} \quad \lim_{k\to\infty} \frac{\sum_{p \in B_k \cap W_\rho B_k(A_k)} \mu(V_\rho^{1} B_k(p))}{\mu(W_\rho B_k(A_k))} = 1.$$

Proof. We have $A \subseteq W_\rho B_k(A) \subseteq \beta^\rho_{2\rho_H(B_k)}(A)$. Thus, $\mu(A) \le \mu\left(W_\rho B_k(A)\right) \le \mu\left(\beta^\rho_{2\rho_H(B_k)}(A)\right)$, where $\beta^\rho_r(A)$ is the r-neighbourhood of a set $A \subseteq S_\rho$. Due to normality of metric spaces, the outer regularity of μ, and the shrinking property of B_k, the last term tends to $\mu(A)$, hence the middle one also does. □

Lemma 2. *Let μ be a Borel measure on S_ρ and let B_k be a sequence that is shrinking and uniformly eating Voronoi boundaries with respect to μ. Suppose that the following condition is satisfied:*

$$\lim_{k\to\infty} |B_k| \sup_{x \in B_k} \mu\left(V_\rho^1 B_k(x)\right) = \mu(S_\rho).$$

Then the empirical measures associated with B_k converge weakly to the normalized measure μ.

Let $\sharp(B)_\rho(A)$ be the number of Voronoi cells of finite set $B \subseteq S_\rho$, which intersects the set $A \subseteq S_\rho$.

Proof. Let $\Delta_k = \sup_{x \in B_k} \mu(V_\rho^1 B_k(x))$, $D_k(\delta) = \{p \in B_k | \mu(V_\rho^1 B_k(p)) \le \delta\}$. Then

$$\mu(S_\rho) \le \delta |D_k(\delta)| + (|B_k| - |D_k(\delta)|)\Delta_k \le |B_k|\Delta_k.$$

For $\tau \in (0,1)$, $\delta_k = \tau\Delta_k$, we obtain $\lim_{k\to\infty} |D_k(\tau\Delta_k)|\Delta_k = 0$. Since $\mu(W_\rho B_k(D_k(\tau\Delta_k))) \to 0$, for any closed set A we have

$$\left| \frac{\mu(A)}{\lim\limits_{k\to\infty} \Delta_k \sharp(B_k)_\rho(A)} - 1 \right| \le 1 - \tau.$$

By the arbitrariness of τ, we get $\mu(A) = \lim_{k\to\infty} \Delta_k \sharp(B_k)_\rho(A)$. Normalization gives the equation $\frac{\mu(A)}{\mu(S_\rho)} = \lim_{k\to\infty} \frac{\sharp(B_k)_\rho(A)}{|B_k|}$. Using that $|A \cap B_k| \le \sharp(B_k)_\rho(A)$, we get the inequality $\limsup_{k\to\infty} \frac{|B_k \cap A|}{|B_k|} \le \frac{\mu(A)}{\mu(S_\rho)}$, which is equivalent to weak convergence (see [1]). □

3. The Class of UAOCE Metrics

Definition 4. A family $\{d_q\}_{q \in M}$ of metrics on open subsets of topological manifold M is called uniformly asymptotically open convex Euclidean (UAOCE) with respect to a auxiliary metric h, which is consistent with the topology of M, if there exists a $\delta_0 \in \mathbb{R}^+$ such that each d_q is a metric on an open neighborhood of the closure of a ball $\mathbb{B}^h_{\delta_0}(q)$ and isometric to the metric of an open convex subset of an Euclidean space, and the following condition is satisfied:

$$\lim_{\delta\to 0^+} \sup_{\substack{h(q,p)\le\delta \\ x,y\in\mathbb{B}^h_\delta(p)\cap\mathbb{B}^h_\delta(q)}} \left|\frac{d_p(x,y)}{d_q(x,y)}-1\right| = 0.$$

Observation 1. The definition of UAOCE is invariant under uniformly equivalent changes of h. Thus, our choice of a specific auxiliary metric is irrelevant as long as it induces the same uniform structure on M. So, if M is compact, we can forget about h.

Observation 2. For each function $f > 0$ such that $\ln f$ is uniformly continuous with respect to h and each UAOCE family of metrics $\{d_q\}_{q\in M}$ with respect to an auxiliary metric h, the family $\{f(q)d_q\}_{q\in M} = \{(fd)_q\}_{q\in M}$ is also UAOCE family with respect to h. This is called a conformal rescaling of a UAOCE family. If two families d and t with respect to h are uniformly asymptotically equivalent in the sense that

$$\lim_{\delta\to 0^+} \sup_{\substack{q\in M \\ x,y\in\mathbb{B}^h_\delta(q)}} \left|\frac{d_q(x,y)}{t_q(x,y)}-1\right| = 0,$$

then their rescalings fd and ft are also.

Observation 3. There is an analogue of the Hausdorff measure for a UAOCE family. Let $A \subseteq \mathbb{B}^h_\delta(q) \subseteq K \subseteq S_{d_q}$ (where K is compact and $\delta \ll \delta_0$) and let B_k be a shrinking sequence in the metric $d_q|_{K^2}$. Set

$$\mathcal{H}^k_d(A) = \lim_{k\to\infty} \sum_{x\in B_k} \mathcal{H}^k_{d_x}(A \cap V^1_{d_q|_{K^2}} B_k(x)).$$

This number does not depend on our choice of q, K, B_k. We extend this premeasure to a measure $\mathcal{H}^k_d$. This is invariant under uniformly asymptotically equivalent changes of $\{d_q\}_{q\in M}$. It is easy to check that

$$\mathcal{H}^k_{fd} = \int f^k d\mathcal{H}^k_d.$$

Definition 5. A metric ρ on M is called a UAOCE with respect to an UAOCE family $\{d_q\}_{q\in M}$ with respect to a auxiliary metric h if it satisfies the following condition:

$$\lim_{\delta\to 0^+} \sup_{\substack{q\in M \\ x,y\in\mathbb{B}^h_\delta(q)}} \left|\frac{\rho(x,y)}{d_q(x,y)}-1\right| = 0.$$

Observation 4. Given a UAOCE metric ρ_1 with respect to $\{d_q\}_{q\in M}$, a metric ρ_2 is a UAOCE metric with respect to $\{d_q\}_{q\in M}$ iff ρ_1 and ρ_2 are uniformly asymptotically equivalent:

$$\lim_{\delta\to 0^+} \sup_{h(x,y)\le\delta} \left|\frac{\rho_1(x,y)}{\rho_2(x,y)}-1\right| = 0.$$

Observation 5. Due to a Lipschitz-like behavior of an UAOCE metric ρ with respect to the family d, our analogue of the Hausdorff measure is equal to the Hausdorff measure of the UAOCE metric:

$$\mathcal{H}^k_d = \mathcal{H}^k_\rho.$$

Lemma 3. *For any UAOCE family $\{d_q\}_{q\in M}$ on a connected space M, there exists a geodesic metric $d'(x,y) \le +\infty$ defined as*

$$d'(x,y) = \lim_{\epsilon\to 0^+} \inf_{\substack{h(q_i,q_{i+1})\le\epsilon \\ q_0=x \\ q_p=y}} \sum_{k=0}^{p-1} d_{q_i}(q_i,q_{i+1}),$$

which is a UAOCE family with respect to the family $\{d_q\}_{q\in M}$. *There exists a constant* $r_0 \in \mathbb{R}^+$ *such that* $h(x,y) \le r_0 \Rightarrow d'(x,y) < +\infty$.

Proof. By the subadditivity of the infimum and by choosing y as an intermediate point in the chain, we get $d'(x,z) \le d'(x,y) + d'(y,z)$.

Using the UAOCE family condition, for each $\varepsilon \in (0,1)$ we find $\epsilon \in (0,\delta_0)$ such that

$$(1-\varepsilon)d_{q_i}(q_i,q_{i+1}) \le d_{q_{i+1}}(q_i,q_{i+1}) \le (1+\varepsilon)d_{q_i}(q_i,q_{i+1}),$$

which yields $d'(x,y) = d'(y,x)$. Using the triangle inequality and the convexity property of d_q , for each $\varepsilon \in (0,1)$ we can find $\delta \in (0,\delta_0)$ such that

$$x,y \in \mathbb{B}^h_\delta(q) \Rightarrow (1-\varepsilon)d_q(x,y) \le d'(x,y) \le (1+\varepsilon)d_q(x,y) < +\infty,$$

which completes the proof. □

Lemma 4. *For any UAOCE family* d *on* $M \simeq \mathbb{S}^1$, d' *is isometric to flat circle.*

Proof. Let $f\colon [0,\mathcal{H}^1_d(M)) \to M$ be a parametrization of the circle. Then $\varphi(t) = \mathcal{H}^1_d(f[[0,t]])$ is a continuous monotone bijection of $[0,\mathcal{H}^1_d(M)]$, i.e., a homeomorphism. So $R = \varphi^{-1}\circ f$ is a normal parametrization, it is Lipschitz continuous at $d_{R(x)}$ with constants $1-\varepsilon$ and $1+\varepsilon$ by the definition of $\mathcal{H}^1_d$. This implies that the length of local curves is preserved. It is easy to check that this implies the isometry of d' and the metric of $\mathbb{R}/\mathcal{H}^1_d(M)\mathbb{Z}$ (the flat circle). □

4. Approximation of Voronoi Cells in $\mathbb{R}^n$

Let $H^a_\rho(x,y) = \{p \in S_\rho | \rho(x,p) \le a\rho(y,p)\}$ be the fluctuated half-space.

Let t be a metric on $\mathbb{R}^n$ induced by the Euclidean norm $\|\cdot\|$, $B \subseteq \mathbb{R}^n \setminus \{0\}$, and $a,\delta \in \mathbb{R}^+$. We aim to estimate $V^a_t B(0) \cap \mathbb{B}^t_\delta(0)$ via $cV^1_t B(0)$ for some constant $c \in \mathbb{R}^+$. If $a = 1$, then $c = 1$. For $a > 1$, a lower bound is $b = c$. For $a < 1$, an upper is $c = 1$.

The inequality $\|p\| \le a\|p-b\|$ for $\|p\| < \delta$, which defines $V^a_t B(0)$, is equivalent to

$$\frac{a^2}{a^2-1}\|b\|^2 \le (a^2-1)\left\|p - \frac{a^2 b}{a^2-1}\right\|^2 .$$

Thus, for any $a > 1$, $V^a_t B(0)$ is the intersection of complements of balls, and for any $a < 1$, it is the intersection of balls. For any $\delta \le C\tilde{t}(0,B)$, we can find a half-space $H^1_t\left(0, \left(\frac{a^2-1}{a^2}C^2+1\right)b\right)$, which represents a scaling of the half-space $H^1_t(0,b)$ by the factor $c = \frac{a^2-1}{a^2}C^2 + 1$. Using this, we obtain two estimates for the measure of Voronoi cells for $a \ge 1$:

$$\mathcal{H}^n_t(V^1_t B(0)) \le \mathcal{H}^n_t(V^a_t B(0) \cap \mathbb{B}^t_\delta(0)) \le \left(\frac{a^2-1}{a^2}C^2+1\right)^n \mathcal{H}^n_t(V^1_t B(0)),$$

for $a \le 1$ the inequalities are reversed.

5. Approximation of Voronoi Cells on UAOCE

Let $\mathcal{L}_\rho B(x) = \{p \in B | V^1_\rho B(p) \setminus V^1_\rho(B \cup \{x\})(p) \ne \varnothing\}$ be the set of adjacent points of x in $B \subseteq S_\rho$. Let ρ be a UAOCE metric. Consider half-spaces within the balls $\mathbb{B}^\rho_\delta(q)$. For sufficiently dense $\{x\} \cup B$, the following inclusions hold:

$$V^{\frac{1-\varepsilon}{1+\varepsilon}}_{d_q}\left(\mathcal{L}_\rho B(x)\right)(x) \subseteq V^{1,0}_\rho B(x) \subseteq V^1_\rho B(x) \subseteq \mathbb{B}^\rho_\delta(x) \cap V^{\frac{1+\varepsilon}{1-\varepsilon}}_{d_x}\left(\mathcal{L}_\rho B(x)\right)(x).$$

This leads to the following estimate of measures:

$$\mathcal{H}^n_{d_x}(V^{\frac{1-\varepsilon}{1+\varepsilon}}_{d_q}\left(\mathcal{L}_\rho B(x)\right)(x)) \le \mathcal{H}^n_{d_x}(V^{1,0}_\rho B(x)) \le \mathcal{H}^n_{d_x}(V^1_\rho B(x)) \le \mathcal{H}^n_{d_x}(\mathbb{B}^\rho_\delta(x) \cap V^{\frac{1+\varepsilon}{1-\varepsilon}}_{d_x}\left(\mathcal{L}_\rho B(x)\right)(x)).$$

By enclosing the Voronoi cell within Euclidean Voronoi cells with specific coefficients and applying the previously derived measure estimates (for $\delta \leq C\tilde{d_x}(x, \mathcal{L}_\rho B(x))$), we obtain:

$$\left(1 - \frac{4C\varepsilon}{(1-\varepsilon)^2}\right)^n \mathcal{H}^n_{d_x}(V^1_{d_q}(\mathcal{L}_\rho B(x))(x)) \leq \mathcal{H}^n_{d_x}(V^{1,0}_\rho B(x))$$

$$\leq \mathcal{H}^n_{d_x}(V^1_\rho B(x)) \leq \left(1 + \frac{4C\varepsilon}{(1+\varepsilon)^2}\right)^n \mathcal{H}^n_{d_x}(V^1_{d_q}(\mathcal{L}_\rho B(x))(x)).$$

For a shrinking sequence B_k and $A \in \mathbb{R}^+$ we have

$$\lim_{k\to\infty} \sup_{\substack{\rho_H(B_k) \leq A\tilde{\rho}(x, B_k \setminus \{x\}) \\ x \in B_k}} \left| \frac{\mathcal{H}^n_{d_x}(V^1_{d_x}(\mathcal{L}_\rho B_k(x))(x))}{\mathcal{H}^n_\rho(V^1_\rho B_k(x))} - 1 \right| = 0.$$

For open cells the limit is similar. This means that each shrinking sequence B_k is uniformly eating Voronoi boundaries with respect to $\mathcal{H}^n_d$.

6. The One-Dimensional Connected Case

Every connected compact 1-manifold (assumed here to be without boundary) is homeomorphic to a circle.

Theorem 1. *Let ρ be a UAOCE metric on a circle M . For any sequence of sets $B_k \in X^\rho_k$, their sequence of empirical measures $\frac{1}{k}\sum_{b\in B_k}\delta_b$ converges weakly to the normalized one-dimensional Hausdorff measure $\mathcal{H}^1_d$ as $k \to \infty$.*

Let $\varepsilon_k(\rho) = \inf_{|A|\leq k} \rho_H(A, S_\rho) > 0$ be Hausdorff optimals and X^ρ_k be the set of a sets of cardinality k with the Hausdorff distance to whole space $\varepsilon_k(\rho)$. It is easy to show that $X^\rho_k \neq \varnothing$ and $\lim_{k\to\infty} \varepsilon_k(\rho) = 0$ on infinite compacta without isolated points.

Proof. Since d' is isometric to a flat circle, $\varepsilon_n(d') = \frac{\mathcal{H}^1_d(M)}{2n}$ (n points with the coordinates $\frac{k\mathcal{H}^1_d(M)}{n}$, $k \in \{0, \ldots, n-1\}$) and $\lim_{k\to\infty} \frac{\varepsilon_k(\rho)}{\varepsilon_k(d')} = 1$, for $B_k \in X^\rho_k$:

$$\lim_{k\to\infty} 2|B_k|(d')_H(B_k) = \lim_{k\to\infty} 2k\frac{(d')_H(B_k)}{\rho_H(B_k)}\frac{\varepsilon_k(\rho)}{\varepsilon_k(d')}\varepsilon_k(d') = \mathcal{H}^1_d(M).$$

It is easy to check that $\mathcal{H}^1_d(V^1_{d'}B_k(x)) \leq 2d'_H(B_k)$, therefore,

$$\mathcal{H}^1_d(M) \leq |B_k| \sup_{x\in B_k} \mathcal{H}^1_d(V^1_{d'}B_k(x)) \leq 2|B_k|d'_H(B_k).$$

This yields that $\lim_{k\to\infty} |B_k| \sup_{x\in B_k} \mathcal{H}^1_d(V^1_{d'}B_k(x)) = \mathcal{H}^1_d(M)$. Using the limit from the previous section, we can transform $V^1_{d'}$ to V^1_ρ . Given that B_k is shrinking and eating Voronoi boundaries, the empirical measures converge weakly to the normalized measure $\mathcal{H}^1_d$. □

7. Remarks

Remark 1. This result can be naturally extended to intervals of bounded length in a UAOCE metric (we should extend the interval to a circle).

Remark 2. Our approach extends smooth flat result to a broader class of compacta where the metric is only asymptotically Euclidean.

Remark 3. While the one-dimensional case yields a clear result due to the simple structure of Voronoi boundaries (which consist of at most two points), the two-dimensional case remains an open problem for the proposed technique. In higher dimensions, the geometry of Voronoi cells $V^1_\rho B_k(b)$ can be significantly more complex.

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References

1 L.B. Koralov, Ya.G. Sinai, Theory of Probability and Random Processes, Springer-Verlag Berlin Heidelberg, 2007, 109-110 p.
2 V.I. Bogachev, Measure Theory, V. 1,2. Springer-Verlag, Berlin, 2007.

Хаусдорф арақашықтығын минимизациялайтын жиындар үшін эмпирикалық өлшемдердің әлсіз жинақталуы туралы

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Аннотация. Жұмыста компакты метрикалық кеңістікке дейінгі Хаусдорф арақашықтығын минимизациялайтын ақырлы жиындардың асимптотикалық өзгерісі зерттеледі. Негізгі мақсат – осы оптималды жиындарда анықталған эмпирикалық өлшемдердің нормаланған Хаусдорф өлшеміне әлсіз жинақталуын жүзеге асыратын шарттарды анықтау. Осы мақсатта бірқалыпты асимптотикалық ашық дөңес евклидтік метрикалар класы енгізіледі. Вороной ұяшықтарының геометриясын қатаң талдауға мүмкіндік беретін бұл метрикалар кеңістіктердің асимптотикалық мағынада локалды евклидтік құрылымға ие екенін сипаттайды. Регулярлы өлшемге қатысты Вороной ұяшықтарының шекараларын бірқалыпты «жұтатын» жиындар ұғымына негізделген әлсіз жинақталудың жеткілікті қайта тұжырымдамасы ұсынылады. Байланысты бірөлшемді көпбейнелердің (шеңбердің) дербес жағдайы үшін нормаланған бірөлшемді Хаусдорф өлшеміне әлсіз жинақталудың толық дәлелі келтіріледі. Бірөлшемді жағдай Вороной ұяшықтары шекараларының қарапайым құрылымына байланысты айқын нәтиже бергенімен де көпөлшемді жағдайлар сәйкес Вороной ұяшықтарының геометриялық күрделілігінің артуына байланысты әлі де ашық мәселе болып қалатыны атап өтіледі.
Түйін сөздер: Хаусдорф қашықтығы, эмпирикалық өлшемдер, кванттау, Вороной ұяшықтары, әлсіз жинақталу, метрикалық компактар.

О слабой сходимости эмпирических мер для множеств, минимизирующих расстояние Хаусдорфа

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Аннотация. В данной работе исследуется асимптотическое поведение конечных множеств, минимизирующих расстояние Хаусдорфа до компактного метрического пространства. Основная цель состоит в установлении условий, при которых эмпирические меры, поддержанные на этих оптимальных множествах, слабо сходятся к нормированной мере Хаусдорфа. Для этого вводится класс равномерно асимптотически открытых выпуклых евклидовых метрик. Эти метрики характеризуют пространства, обладающие локальной евклидовой структурой в асимптотическом смысле, что позволяет проводить строгий анализ геометрии ячеек Вороного. В работе предлагается достаточная переформулировка слабой сходимости, основанная на концепции множеств, равномерно «поглощающих» границы ячеек Вороного относительно регулярной меры. Для частного случая связного одномерного многообразия (окружности) приводится полное доказательство слабой сходимости к нормированной одномерной мере Хаусдорфа. В сравнении с одномерным случаем, где даётся ясный результат благодаря простой структуре границ ячеек Вороного, в заключении отмечается, что случаи большей размерности остаются открытой задачей из-за возросшей геометрической сложности соответствующих ячеек Вороного.

Ключевые слова: расстояние Хаусдорфа, эмпирические меры, квантование, ячейки Вороного, слабая сходимость, метрические компакты.

References

1 L.B. Koralov, Ya.G. Sinai, Theory of Probability and Random Processes, Springer-Verlag Berlin Heidelberg, 2007, 109-110 p.
2 V.I. Bogachev, Measure Theory, V. 1,2. Springer-Verlag, Berlin, 2007.

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