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GENERALIZATIONS OF THE RUDIN–KEISLER PREORDER AND THEIR MODEL-THEORETIC APPLICATIONS

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Abstract. Generalizing of the Rudin–Keisler preorder, we introduce relations R_α (and $R_{<\alpha}$) on the set $\boldsymbol{\beta}\omega$ of ultrafilters on ω. They form an ordinal sequence of length ω_1 which is strictly increasing by inclusion and lies between the Rudin–Keisler preorder and the Comfort preorder. We show that the composition of these relations is expressed via a multiplication-like operation on ordinals. Explicit calculations of this operation show that $R_{<\alpha}$ is transitive (and so, a preorder) iff the ordinal α is multiplicatively indecomposable. The proposed constructions have several model-theoretic consequences. Generalizing significantly results of García-Ferreira, Hindman, and Strauss concerning an interplay between ultrafilter extensions of semigroups and the Comfort preorder, we prove that for every model $\mathfrak{A}$, ultrafilter $\mathfrak{u}$, and ordinal α, the set $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ forms a submodel of the ultrafilter extension $\boldsymbol{\beta}\mathfrak{A}$ of $\mathfrak{A}$ iff the ordinal α is additively indecomposable. Furthermore, generalizing Blass' characterization of the Rudin–Keisler preorder via ultrapowers, we characterize the relations R_α, and in particular, the Comfort preorder, via a specific version of limit ultrapowers.

Keywords: ultrafilter, ultrafilter extension, Rudin–Keisler preorder, Comfort preorder, ultrapower, limit ultrapower

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1. INTRODUCTION

As usual, $\boldsymbol{\beta}X$ denotes the standard Čech–Stone compactification of the discrete space X, which we identify with the set of ultrafilters over X (see [2, 7]). We consider here ultrafilters over ω although most of our results remain true for ultrafilters over any infinite set X.

Since the 1960s natural preorders have been studied in the theory of ultrafilters, of which the Rudin–Keisler preorder and the Comfort preorder are well known. The *Rudin–Keisler* preorder $\le_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$ is defined by letting $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}$ iff there exists $f : \omega \to \omega$ such that $\widetilde{f}(\mathfrak{v}) = \mathfrak{u}$, where $\widetilde{f} : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ is the continuous extension of f. The *Comfort* preorder $\le_{\mathrm{C}}$ on $\boldsymbol{\beta}\omega$ is defined by letting $\mathfrak{u} \le_{\mathrm{C}} \mathfrak{v}$ iff any $\mathfrak{v}$-compact space is $\mathfrak{u}$-compact, where a space X is $\mathfrak{u}$*-compact* iff $\widetilde{f}(\mathfrak{u}) \in X$ for any $f : \omega \to X$. (See [2, 7] for more on ultrafilters and $\le_{\mathrm{RK}}$, and [3, 4] for $\le_{\mathrm{C}}$.)

We introduce an ordinal sequence of relations lying between these two preorders and prove a number of their properties, including model-theoretic characterizations.

2. MAIN RESULTS

2.1. Basic definitions

For each ordinal α, we define a binary relation R_α on $\boldsymbol{\beta}\omega$ as follows.

Definition 1. For $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$ and ordinal $\alpha > 0$, let:

(i) $\mathfrak{u}\, R_0\, \mathfrak{v}$ iff $\mathfrak{u}$ is principal,

(ii) $R_{<\alpha} := \bigcup_{\beta<\alpha} R_\beta$,

(iii) $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous map $f : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ such that

  (a) $f(\mathfrak{v}) = \mathfrak{u}$ and

  (b) $f(n)\, R_{<\alpha}\, \mathfrak{v}$ for all $n < \omega$.

The hierarchy is non-degenerate and lies between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ as stated in two following theorems.

Theorem 1. *For all* α,

(i) $R_1 = \leq_{\mathrm{RK}}$ *;*

(ii) $R_{<(\alpha+1)} = R_\alpha$ *;*

(iii) $R_{<\alpha} \subset R_\alpha$ *(the strict inclusion) whenever* $\alpha < \omega_1$ *;*

(iv) $R_{<\omega_1} = R_{\omega_1}$.

Theorem 2. $R_{<\omega_1} = R_{\omega_1} = \leq_{\mathrm{C}}$. *So we have the increasing chain:*

$$R_0 \subset R_1 = \leq_{\mathrm{RK}} \subset R_2 \subset \ldots \subset R_{<\omega} \subset R_\omega \subset R_{\omega+1} \subset \ldots$$
$$\subset R_{<\alpha} \subset R_\alpha \subset R_{\alpha+1} \subset \ldots \subset R_{<\omega_1} = R_{\omega_1} = \leq_{\mathrm{C}}.$$

The R_α relations can also be defined in another way. If X, Y are spaces and α is an ordinal, $f : X^\alpha \to Y$ is *right-continuous w.r.t.* $A \subseteq X$ iff for all $\beta < \alpha$ the shift $x \mapsto f(a_0, a_1, \ldots, x, b_{\beta+1}, b_{\beta+2}, \ldots)$ is continuous whenever $a_0, a_1, \ldots \in A$ and $b_{\beta+1}, b_{\beta+2}, \ldots \in X$. As shown in [9, 10], if $n < \omega$, X is discrete, and Y is compact Hausdorff, then every $f : X^n \to Y$ uniquely extends to $\widetilde{f} : (\boldsymbol{\beta}X)^n \to Y$ that is right-continuous w.r.t. X. The ultrafilter extensions of relations $P \subseteq X^n$ can be defined in a similar manner. This provides a canonical way to obtain, for an arbitrary first-order model $\mathfrak{A}$, its *ultrafilter extension* $\boldsymbol{\beta}\mathfrak{A}$ ([9, 10]), generalizing the well-known construction of ultrafilter extensions of semigroups comprehensively treated in [7]. (Some historical remarks can be found in [8].)

If $n < \omega$, the relations R_n can be redefined in terms of ultrafilter extensions of n-ary operations on ω as follows: $\mathfrak{u}\, R_n\, \mathfrak{v}$ iff there exists $f : \omega^n \to \omega$ such that $\widetilde{f}(\mathfrak{v}, \ldots, \mathfrak{v}) = \mathfrak{u}$. These observations can be expanded to all R_α by using ω-ary operations on ω. Such an operation is identified with a continuous map of the Baire space ω^ω into the discrete space ω; these maps admit a natural hierarchy ranked by countable ordinals.

Proposition 1. *Any continuous (in the Baire space)* $f : \omega^\omega \to \omega$ *uniquely extends to* $\widetilde{f} : (\boldsymbol{\beta}\omega)^\omega \to \boldsymbol{\beta}\omega$ *that is right-continuous w.r.t.* ω *(in other words,* ω*-ary operations on* ω *extend to such operations on* $\boldsymbol{\beta}\omega$*).*

Proposition 2. *Let* $\alpha < \omega_1$ *and* $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$. *Then* $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ *iff there exists a continuous* $f : \omega^\omega \to \omega$ *of rank* α *such that* $\widetilde{f}(\mathfrak{v}, \mathfrak{v}, \ldots) = \mathfrak{u}$.

2.2. Composition of $\mathbf{R}_\alpha$ and the relations $\leq_\alpha$

Using the alternative definition of the R_α relations, it is easy to see that for all $n, m < \omega$

$$R_m \circ R_n = R_{nm}$$

(so R_n are not preorders for $2 \leq n < \omega$). The composition of arbitrary R_α (or $R_{<\alpha}$) is expressed via a multiplication-like operation on ordinals. To simplify notation, denote $\sup_{\gamma<\alpha}(\gamma \cdot \beta)$ by

$(<\alpha)\cdot\beta$; the explicit calculation of these ordinals, used in getting the following result, is rather cumbersome.

Theorem 3. *Let* $\alpha,\beta<\omega_1$.

(i) $R_\alpha\circ R_\beta=R_\gamma$ *where*

$$\gamma=\begin{cases}\beta\cdot\alpha & \text{if } \beta=0 \text{ or } \alpha<\omega,\\ \beta\cdot(\alpha+1)-1 & \text{if } 0<\beta<\omega \text{ and } \alpha\geq\omega,\\ \beta\cdot(\alpha+1) & \text{if } \alpha,\beta\geq\omega;\end{cases}$$

(ii) *If* $\alpha>0$ *is limit, then* $R_{<\alpha}\circ R_\beta=R_{<\gamma}$ *where* $\gamma=\beta\cdot\alpha$;

(iii) *If* $\beta>0$ *is limit, then* $R_\alpha\circ R_{<\beta}=R_{<\gamma}$ *where*

$$\gamma=\begin{cases}(<\beta)\cdot\alpha & \text{if } \alpha<\omega,\\ (<\beta)\cdot(\alpha+1) & \text{otherwise};\end{cases}$$

(iv) *If* $\alpha,\beta>0$ *are limit, then* $R_{<\alpha}\circ R_{<\beta}=R_{<\gamma}$ *where* $\gamma=(<\beta)\cdot\alpha$.

Corollary 1. Let $2\leq\alpha\leq\omega_1$. Then $R_{<\alpha}$ is a preorder iff α is multiplicatively indecomposable.

Define preorders between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ by letting $\leq_0\,=\,\leq_{\mathrm{RK}}$ and $\leq_{1+\alpha}\,=\,R_{<\omega^{\omega^\alpha}}$ for all $\alpha\leq\omega_1$. So, if α is infinite, $R_{<\alpha}=\,\leq_\alpha$ iff α is an epsilon number. Also $\leq_\alpha\circ\leq_\beta\,=\,\leq_\gamma$ where $\gamma=\max(\alpha,\beta)$.

2.3. Model-theoretic applications

In this section, we will examine two applications in model theory.

The first concerns ultrafilter extensions. As shown in [5], for any ultrafilter $\mathfrak{v}$ and semigroup S, the set $\{\mathfrak{u}:\mathfrak{u}\leq_{\mathrm{C}}\mathfrak{v}\}$ forms a subsemigroup of $\boldsymbol{\beta}S$. This can be expanded to arbitrary first-order models and relations $R_{<\alpha}$ as follows.

Corollary 2. For all ordinals $\alpha>1$, ultrafilters $\mathfrak{v}$, and models $\mathfrak{A}$, the following are equivalent:

(i) $\{\mathfrak{u}:\mathfrak{u}\,R_{<\alpha}\,\mathfrak{v}\}$ forms a submodel of the model $\boldsymbol{\beta}\mathfrak{A}$;

(ii) α is additively indecomposable.

Consequently, for all $\alpha>0$, $\mathfrak{v}$, and $\mathfrak{A}$, the set $\{\mathfrak{u}:\mathfrak{u}\leq_\alpha\mathfrak{v}\}$ forms a submodel of $\boldsymbol{\beta}\mathfrak{A}$.

The second application concerns ultrapowers. For a model $\mathfrak{A}$ with the universe A, a set I, an ultrafilter $\mathfrak{u}$ over I, and $f:I\to A$, let $f_\mathfrak{u}$ denote the $\mathfrak{u}$-equivalence class of f, and $\prod_\mathfrak{u}\mathfrak{A}$ the ultrapower of $\mathfrak{A}$ by $\mathfrak{u}$. As shown in [1], the Rudin–Keisler preorder has a natural characterization in terms of ultrapowers: $\mathfrak{u}\leq_{\mathrm{RK}}\mathfrak{v}$ iff $\prod_\mathfrak{u}\mathfrak{A}\preceq\prod_\mathfrak{v}\mathfrak{A}$ for all models $\mathfrak{A}$, and also iff $\prod_\mathfrak{u}\mathfrak{N}\preceq\prod_\mathfrak{v}\mathfrak{N}$ where $\mathfrak{N}$ is the *full model of* ω, i.e., it has the universe ω and all relations and operations on ω.

It is not difficult to characterize R_n with $n<\omega$ via ultrapowers in a similar manner:

Proposition 3. *For all* $\mathfrak{u},\mathfrak{v}$, *and* $n<\omega$, *the following are equivalent:*

(i) $\mathfrak{u}\,R_n\,\mathfrak{v}$;

(ii) $\prod_\mathfrak{u}\mathfrak{A}\preceq\prod_\mathfrak{v}\ldots\prod_\mathfrak{v}\mathfrak{A}$ *(n times) for all models* $\mathfrak{A}$;

(iii) $\prod_\mathfrak{u}\mathfrak{N}\preceq\prod_\mathfrak{v}\ldots\prod_\mathfrak{v}\mathfrak{N}$ *(n times).*

To handle the case $\alpha\geq\omega$, the following "skew version" of limit ultrapowers is used. First for any $e:A\to B$ define $e^\mathfrak{u}:\prod_\mathfrak{u}A\to\prod_\mathfrak{u}B$ by letting $e^\mathfrak{u}(g_\mathfrak{u}):=(e\circ g)_\mathfrak{u}$. Clearly, $e:\mathfrak{A}\preceq\mathfrak{B}$ implies $e^\mathfrak{u}:\prod_\mathfrak{u}\mathfrak{A}\preceq\prod_\mathfrak{u}\mathfrak{B}$. Then for every model $\mathfrak{A}$, ultrafilter $\mathfrak{u}$, and ordinals α, define the models $\mathfrak{A}_{\mathfrak{u},\alpha}$ and their embeddings $e_{\beta\alpha}$ for $\beta<\alpha$:

(i) $\mathfrak{A}_{\mathfrak{u},0}:=\mathfrak{A}$, $\mathfrak{A}_{\mathfrak{u},1}:=\prod_\mathfrak{u}\mathfrak{A}$, and $e_{01}=d$ (the diagonal map);

(ii) if α is limit, then $\mathfrak{A}_{\mathfrak{u},\alpha}:=\lim_{\beta\to\alpha}\mathfrak{A}_{\mathfrak{u},\beta}$ w.r.t. the system $\{e_{\gamma\beta}\}_{\gamma<\beta<\alpha}$, and the maps $e_{\beta\alpha}$ ($\beta<\alpha$) are defined naturally;

(iii) if $\alpha=\beta+1$, then $\mathfrak{A}_{\mathfrak{u},\alpha}:=\prod_\mathfrak{u}\mathfrak{A}_{\mathfrak{u},\beta}$, and the maps $e_{\delta\alpha}$ ($\delta<\alpha$) are defined as follows:

  (a) if $\beta=\gamma+1$, then $e_{\beta\alpha}:=e^\mathfrak{u}_{\gamma\beta}$;

(b) if $\beta > 0$ is limit, then for any $g \in A_{\mathfrak{u},\beta}$

$$e_{\beta\alpha}(g) := e^{\mathfrak{u}}_{\gamma\beta}(h)$$

for some $\gamma < \beta$ and $h \in g \cap A_{\mathfrak{u},\gamma+1}$;

(c) for any $\delta < \beta$, $e_{\delta\alpha} := e_{\beta\alpha} \circ e_{\delta\beta}$.

Lemma 1. *All $\mathfrak{A}_{\mathfrak{u},\alpha}$ are well defined, and if $\beta < \alpha$ then $e_{\beta\alpha} : \mathfrak{A}_{\mathfrak{u},\beta} \preceq \mathfrak{A}_{\mathfrak{u},\alpha}$.*

Theorem 4. *For $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$, and a limit ordinal $\alpha > 0$, the following are equivalent:*

(i) $\mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}$;

(ii) *there is $\beta < \alpha$ such that $\mathfrak{A}_{\mathfrak{u},1} \preceq \mathfrak{A}_{\mathfrak{v},\beta}$ for all models $\mathfrak{A}$;*

(iii) *there is $\beta < \alpha$ such that $\mathfrak{N}_{\mathfrak{u},1} \preceq \mathfrak{N}_{\mathfrak{v},\beta}$.*

Consequently, $\mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v}$ iff $\mathfrak{A}_{\mathfrak{u},1} \preceq \mathfrak{A}_{\mathfrak{v},\omega_1}$ for all models $\mathfrak{A}$, and also iff $\mathfrak{N}_{\mathfrak{u},1} \preceq \mathfrak{N}_{\mathfrak{v},\omega_1}$.

3. CONCLUSION AND DISCUSSION

The relations R_α provide a natural “smooth” transition from the Rudin-Keisler preorder to the Comfort preorder surprisingly showing that these two preorders are essentially of the same nature. This provides new methods for establishing properties of the Comfort preorder on $\boldsymbol{\beta}\omega$. The proposed skew version of limit ultrapowers raises natural questions. What are the properties of this model-theoretic construction? Is it true that the usual and skew limit ultrapowers are elementarily embedded in each other (perhaps with a rank shift)?

Authors' Contributions: the authors' contribution is equal.

References

1 Blass A.R. Orderings on ultrafilters: PhD Thesis, Harward University, Cambridge, Mass., 1970.
2 Comfort W.W., Negrepontis S. The theory of ultrafilters. Part of the book series: Grundlehren math. Wiss., (GL. Vol. 211) Springer, 1974.
3 García-Ferreira S. Three orderings on $\boldsymbol{\beta}(\omega) \setminus \omega$ // Top. Appl. -1993. -Vol.50. -P.119–216.
4 García-Ferreira S. Comfort types of ultrafilters //Proc. Amer. Math. Soc. -1994. -Vol. 120. -P.1251–1260.
5 García-Ferreira S., Hindman N., Strauss D. Orderings of the Stone–Cech remainder of a discrete semigroup //Top. Appl. -1999. - Vol. 97:1-2. -P.127–148.
6 Goranko V. Filter and ultrafilter extensions of structures: universal-algebraic aspects, tech. report, 2007.
7 Hindman N., Strauss D., Algebra in the Stone–Cech Compactification: Theory and Applications, sec. rev. and ext. ed., De Gruyter, 2012.
8 Poliakov N. L., Saveliev D. I. On ultrafilter extensions of first-order models and ultrafilter interpretations // Arch. Math. Log. -2021. - Vol. 60. -P.625-681.
9 Saveliev D. I. Ultrafilter extensions of models, Log. Appl., Lect. Notes Comp. Sci. Springer, 2011. -P.162–177.
10 Saveliev D. I. On ultrafilter extensions of models, The Infinity Project Proc., CRM Documents, 11, S.-D. Friedman et al. (eds.), 2012. P.599–616.

Рудин-Кейслер алдыңғы ретінің жалпылаулары және олардың теориялық-модельдік қолданыстары

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Аннотация. Мақалада Рудин–Кейслердің алғы ретін жалпылай отырып, ω-дағы $\boldsymbol{\beta}\omega$ ультрасүзгілер жиынында R_α (және $R_{<\alpha}$) қатынастары енгізілді. Бұл қатынастар ω_1 ұзындықты, кірістіру бойынша қатаң өспелі және Рудин-Кейслер және Комфорт алғы реттерінің арасында жататын ординал тізбек құрайды. Мақалада осы қатынастардың композициясы ординалдарды көбейту амалына жақын операция арқылы өрнектелетіні көрсетілді. Бұл операцияны нақты есептеу α ординалы мультипликативті жіктелмесе және тек сонда ғана $R_{<\alpha}$ қатынасы транзитивті екендігін (сондықтан алғы рет те болатындығын) көрсетеді. Ұсынылған құрылымдар бірқатар теориялық-модельдік салдарға ие. Жартылай топтардың ультракеңеюлері мен Комфорттың алғы реті арасындағы өзара байланыс туралы Гарсия-Феррейра, Хиндман және Штраусс нәтижелерін едәуір жалпылай отырып, авторлар мақалада кез келген $\mathfrak{A}$ моделі, $\mathfrak{u}$ ультрасүзгісі мен α ординалы үшін $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ жиыны $\boldsymbol{\beta}\mathfrak{A}$ ультракеңеюінде $\mathfrak{A}$ моделінің ішкі моделін құруы үшін α ординалы аддитивті жіктелмейтін болуы қажетті және жеткілікті екендігі дәлелденді. Сонымен қатар, ультрадәрежелер арқылы Бласс берген Рудин–Кейслер алғы ретінің сипаттамасын жалпылай отырып, авторлармен R_α қатынастарының, дербес жағдайда, шектеулі ультрадәрежелердің арнайы нұсқамалары арқылы Комфорт алғы ретінің сипаттамасы берілді.

Түйін сөздер: ультрасүзгі, ультракеңею, Рудин–Кейслер алғы реті, Комфорт алғы реті, ультрадәреже, шектік ультрадәреже.

Обобщения предпорядка Рудин–Кейслера и их теоретико-модельные приложения

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Аннотация. В статье введены отношения R_α (и $R_{<\alpha}$) на множестве $\boldsymbol{\beta}\omega$ ультрафильтров на ω, которые обобщают предпорядок Рудин-Кейслера. Эти отношения образуют ординальную последовательность длины ω_1, строго возрастающую по включению и лежащую между предпорядком Рудин–Кейслера и предпорядком Комфорта. В статье показано, что композиция этих отношений выражается через операцию, близкую к умножению ординалов. Явные вычисления этой операции показывают, что $R_{<\alpha}$ является транзитивным (и, следовательно, предпорядком), тогда и только тогда, когда ординал α мультипликативно неразложим. Предложенные конструкции имеют ряд теоретико-модельных следствий. Существенно обобщая результаты Гарсия-Ферейры, Хиндмана и Штраусс о взаимосвязи между ультрарасширениями полугрупп и предпорядком Комфорта, авторы доказывают, что для любой модели $\mathfrak{A}$, ультрафильтра $\mathfrak{u}$ и ординала α, множество $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ образует подмодель ультрарасширения $\boldsymbol{\beta}\mathfrak{A}$ модели $\mathfrak{A}$, тогда и только тогда, когда ординал α аддитивно неразложим. Кроме того, обобщая характеризацию предпорядка Рудин–Кейслера, данную Блассом с помощью ультрастепеней, авторы дают характеризацию отношений R_α и, в частности, предпорядка Комфорта через специальную версию предельных ультрастепеней.

Ключевые слова: ультрафильтр, ультрарасширение, предпорядок Рудин–Кейслера, предпорядок Комфорта, ультрастепень, предельная ультрастепень.

References

1 Blass A. R. Orderings on ultrafilters: PhD Thesis, Harward University, Cambridge, Mass., 1970.

2 Comfort W. W., Negrepontis S. The theory of ultrafilters. Part of the book series: Grundlehren math. Wiss., (GL, Vol.211) Springer, 1974.

3 García-Ferreira S. Three orderings on $\boldsymbol{\beta}(\omega) \setminus \omega$, Top. Appl., 1993. V.50, P.119–216.

4 García-Ferreira S. Comfort types of ultrafilters, Proc. Amer. Math. Soc., 1994. V.120, P.1251–1260.

5 García-Ferreira S., Hindman N., Strauss D. Orderings of the Stone–Cech remainder of a discrete semigroup, Top. Appl., 1999. Vol.97:1–2. P.127–148.

6 Goranko V. Filter and ultrafilter extensions of structures: universal-algebraic aspects, tech. report, 2007.

7 Hindman N., Strauss D., Algebra in the Stone–Cech Compactification: Theory and Applications, sec. rev. and ext. ed., De Gruyter, 2012.

8 Poliakov N. L., Saveliev D. I. On ultrafilter extensions of first-order models and ultrafilter interpretations, Arch. Math. Log., 2021. V.60. P.625–681.

9 Saveliev D. I. Ultrafilter extensions of models, Log. Appl., Lect. Notes Comp. Sci., Springer, 2011. P.162–177.
10 Saveliev D. I. On ultrafilter extensions of models, The Infinity Project Proc., CRM Documents, 11, S.-D. Friedman et al. (eds.), 2012. P.599–616.

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