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STRUCTURAL SUMS ON THE COMPLEX PLANE AND THEIR APPLICATION TO COMPOSITE MATERIALS

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Abstract. Consider a two-dimensional, two-component composite consisting of non-overlapping, identical circular disks embedded in a uniform host. The conductivity of the considered composites is modeled by the $\mathbb{R}$-linear conjugation problem for functions analytic in the domains occupied by the components of composites and Hölder continuous in their closures. The effective conductivity of these dispersed random composites can be determined using constructive homogenization theory, which assumes strictly stationary fields. The Eisenstein summation method is applied to analyze conditionally convergent sums that arise during the homogenization process. Additionally, the Clausius-Mossotti approximation, which is valid for macroscopically isotropic two-dimensional composites up to $O(f^3)$, is justified for random composites. A new analytical formula for the effective conductivity tensor of macroscopically anisotropic composites is derived up to $O(f^3)$. This formula contains a conditionally convergent sum S_2' coinciding with the Rayleigh lattice sum for the square array of disks calculated by the Eisenstein summation method. Moreover, S_2' depends on a curve connecting a finite point with infinity. This approach provides a robust framework for understanding the behavior of such materials and offers insights into their effective conductivity properties.

Keywords: structural sum, effective properties of composites, homogenization, strictly stationary field, R-linear problem, asymptotic analysis

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1. Introduction

The theory of measure serves as the theoretical foundation for random structures within the framework of dispersed composites. The randomness of these structures is reflected in the random coefficients of partial differential equations that model the process. A rigorous homogenization theory for elliptic operators with highly oscillated random coefficients has been developed [8, 10, 7, 20]. These coefficients are represented by a symmetric, positively defined matrix in the plane

$$\boldsymbol{\lambda}(\mathbf{x}) = \begin{pmatrix} \lambda_{xx}(\mathbf{x}) & \lambda_{xy}(\mathbf{x}) \\ \lambda_{xy}(\mathbf{x}) & \lambda_{yy}(\mathbf{x}) \end{pmatrix} \tag{1}$$

called the local conductivity tensor. In the theory of random composites, it is assumed that the matrix $\boldsymbol{\lambda}(\mathbf{x})$ is a strictly stationary spatial field.

Figure 1 – Circular inclusions on the complex plane.

For macroscopically isotropic fields, the matrix function (1) simplifies to a scalar function $\lambda(\mathbf{x})$, more precisely,

$$\boldsymbol{\lambda}(\mathbf{x}) = \lambda(\mathbf{x})I, \tag{2}$$

where I stands for the identity matrix.

In the context of two-dimensional (2D) media, it is convenient to consider the measured matrix-function $\boldsymbol{\lambda}(z)$ with the complex argument $z = x_1 + \mathrm{i}x_2 \in \mathbb{C}$, where $x_1, x_2 \in \mathbb{R}^2$. For shortness, the same designation $\boldsymbol{\lambda}(z) \equiv \boldsymbol{\lambda}(x_1, x_2)$ is used for this function. The stationary conductivity on the plane is described by elliptic equations

$$\nabla \cdot (\boldsymbol{\lambda}(x_1, x_2)\nabla u(x_1, x_2)) = 0, \quad (x_1, x_2) \in \mathbb{R}^2. \tag{3}$$

The partial differential equation (3) for an unknown function $u(x_1, x_2)$ is considered in a Sobolev space [10]. Moreover, $u(x_1, x_2) \equiv u(z)$ is bounded at infinity.

In the present paper, we restrict our attention to a two-dimensional, two-component composite made from a collection of non-overlapping, identical, circular disks embedded in an otherwise uniform host. Several researchers have taken a different approach by assuming a specific regular geometry for the composite material. Rayleigh [18] addressed this problem for a regular square array of disks in 1892. He introduced lattice sums for regular structures and highlighted the conditionally convergent lattice sum S_2, proving that $S_2 = \pi$ calculated by the Eisenstein summation [21]. Rayleigh's method was extended to other regular structures by McPhedran et al. in a series of papers [11, 13, 12]. The lattice sums were further extended to random locations of disks in [14, 15].

Besides some special cases like Dykhne-Keller-Mathéron approach, the problem of constructive description of the strictly stationary random field $\boldsymbol{\lambda}(\mathbf{x})$ was considered in [10]. In particular, it was demonstrated that homogenization holds for an almost-periodic matrix $\boldsymbol{\lambda}(\mathbf{x})$ in the sense of Bohr and Bezikovich. More precisely, the components of $\boldsymbol{\lambda}(\mathbf{x})$ are approximated by a sequence of trigonometric polynomials converging to $\boldsymbol{\lambda}(\mathbf{x})$ in a space defined by the Bezikovich norm

$$\|F\|^2 = \lim\sup_{R\to\infty} \frac{1}{\pi R^2} \int_{|\mathbf{x}|\le R} |F(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}. \tag{4}$$

Here, $F \in L^2_{loc}(\mathbb{R}^2)$ belongs to equivalence classes detailed in [10].

A dispersed 2D two-phase composite with equal circular inclusions is described as follows. Consider non-overlapping disks $D_k := \{z \in \mathbb{C} : |z - a_k| < r,\ k = 1, 2, \ldots\}$ on the complex plane shown in Figure 1. Let $D^+ := \cup_{k=0}^{\infty} D_k$ and $D^- := \hat{\mathbb{C}} \backslash (D^+ \cup \partial D^+)$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane, ∂D^+ is the boundary of D^+.

The considered composite forms a strictly stationary field if the countable set of centers

$$\mathcal{A} = \{a_1, a_2, a_3, \ldots\} \tag{5}$$

is strictly stationary and $|a_k - a_m| \ge 2r$ for $k \ne m$. In the present paper, we assume that $|a_k| \ge r$ for all $k = 1, 2, \ldots$. It simplifies the consideration [14], where the set $\mathcal{A}$ contains the point $a_0 = 0$.

In the context of [10] related to ergodicity, we assume that the set of points $\mathcal{A}$ is a typical element of the considered random set. It is assumed that the function $\lambda(z)$ is piecewise constant,

$$\lambda(z) = \begin{cases} \lambda_1 & z \in D^+, \\ \\ 1 & z \in D^-. \end{cases} \tag{6}$$

Here, the conductivity of the host is normalized to unity. Therefore, the conductivity λ_1 can be considered as the ratio of conductivities in the inclusions and in the host.

Let the set $\mathcal{A}$ form a double periodic structure. Infinitesimally small random perturbations of such a structure were studied in [10]. In particular, an extension of the Clausisus-Mossotti (Maxwell) approximation for the effective conductivity was derived. A more general shaken model with no small perturbation was discussed in [3, 4]. A double periodicity condition of perturbation was assumed. Similar model for the security sphere approach was developed in [2].

The effective conductivity of dispersed composites for an arbitrary set of points $\mathcal{A}$ was investigated in [14] by means of the absolutely convergent series associated with the classic elliptic functions by Weierstrass. In the present paper, we continue to discuss this question based on the conditionally convergent Eisenstein series. This approach allows to simplify and extend the results [14].

2. $\mathbb{R}$-linear problem and functional equations

Introduce the dimensionless contrast parameter

$$\varrho = \frac{\lambda_1 - 1}{\lambda_1 + 1}. \tag{7}$$

The complex velocities of the considered problems, the functions $\psi(z)$ and $\psi_k(z)$ ($k = 1, 2, \dots$), are analytic in the domains D^- and D^+, respectively. The function $\psi_k(z)$ is Hölder continuous in $|z - a_k| \leq r$. The function $\psi(z)$ is Hölder continuous in the closure of D^-, except at the point $z = \infty$, where it is bounded. The functions $\psi(z)$ and $\psi_k(z)$ satisfy the $\mathbb{R}$-linear conjugation condition [19]

$$\psi(t) = \psi_k(t) + \varrho r^2 (t - a_k)^{-2}\, \overline{\psi_k(t)} - c(t), \quad |t - a_k| = r \ (k = 1, 2, \dots), \tag{8}$$

where $c(t)$ is a given Hölder continuous function. It is assumed for definiteness that $c(t) = 1$. Here, the complex number $1 + \mathrm{i}0$ models the external vector field $(1, 0)$ applied at infinity. It is worth noting that the problem (8) is stated in the infinitely connected domain contrary to [19], where the problem was considered in a finitely connected domain on the plane or on the plane torus, i.e., in a class of doubly periodic functions.

Boundary value problems for infinitely connected domains frequently require the study of infinite winding number (index) [9]. However, due to the inequality $|\varrho| \leq 1$, its winding number vanishes, and the problem has a unique solution up to an additive arbitrary constant [14, 5].

The theory of the Eisenstein series was developed for doubly periodic functions [21] closely related to the elliptic Weierstrass functions [1]. Consider a double periodic lattice on the complex plane formed by the fundamental translation vectors ω_1 and ω_2. It is convenient to consider these vectors as such complex numbers for which $\omega_1 > 0$, $\mathrm{Im}\,\omega_2 > 0$, and $\omega_1 \mathrm{Im}\,\omega_2 = 1$. The last relation means that the area of the parallelogram constructed on ω_1 and ω_2 holds unity. Let $\omega = m_1\omega_1 + m_2\omega_2$, where m_1 and m_2 run over integer numbers $\mathbb{Z}$. Introduce the Eisenstein-Rayleigh lattice sums [18]

$$S_m = \sum_{m_1, m_2}{}' \omega^{-m}, \quad m = 2, 3, \dots, \tag{9}$$

where the prime means that the pair $(m_1, m_2) = (0, 0)$ is skipped in the summation. The series (9) for $m = 2$ converges conditionally. Therefore, its value depends on the order of summation. In the present paper, the order is fixed by using the Eisenstein summation [21]

$$\sum_{m_1,m_2}{}^{(e)} := \lim_{M_2\to\infty}\lim_{M_1\to\infty}\sum_{p_2=-M_2}^{M_2}\sum_{p_1=-M_1}^{M_1}. \tag{10}$$

The Eisenstein summation is defined by the external field parallel to the real axis [18]. The Eisenstein function is introduced through the series

$$E_m(z) := \sum_{m_1,m_2}{}^{(e)} \frac{1}{(z-\omega)^m}. \tag{11}$$

It is conditionally, and almost uniformly convergent for $m = 2$, and absolutely and uniformly convergent for $m > 2$ [21]. The Eisenstein and elliptic Weierstrass functions are related by the formula

$$E_2(z) = \wp(z) + S_2. \tag{12}$$

The set of points (5) is linearly ordered. It can be doubly ordered. In the case of a doubly periodic set, it can be enumerated as follows $\mathcal{A} = \{m_1\omega_1 + m_2\omega_2 : \ m_1, m_2 \in \mathbb{Z}\}$. The doubly periodic set $\mathcal{A}$ is strictly stationary.

We now proceed to consider non-periodic sets, not assuming their stationarity, and considering other assumptions. First, we assume that the concentration of non-overlapping disks exists and is normalized to $f = \pi r^2$. More precisely, there exists the limit

$$\lim_{n\to\infty}\frac{n}{|D_n^-|} = 1, \tag{13}$$

where a rectangle D_n^- contains n disks and extends to the infinite point, as $n \to \infty$.

The double numeration of $\mathcal{A}$ follows the Eisenstein summation. Consider a point $a_{m_1,m_2} \in \mathcal{A}$ with the nonnegative real and imaginary parts. The subscripts m_1 are ordered by the real part and m_2 by the imaginary part, i.e.,

$$\begin{aligned} &0 \le \mathrm{Re}\ a_{0,m_2} \le \mathrm{Re}\ a_{1,m_2} \le \mathrm{Re}\ a_{2,m_2} \le \ldots, \\ &0 \le \mathrm{Im}\ a_{m_1,0} \le \mathrm{Im}\ a_{m_1,1} \le \mathrm{Im}\ a_{m_1,2} \le \ldots. \end{aligned} \tag{14}$$

The points $a_{m_1,m_2} \in \mathcal{A}$ in other quadrants of the planes are ordered analogously.

The $\mathbb{R}$-linear problem (8) was reduced to the system of functional equations [14]

$$\begin{aligned} \psi_k(z) = \varrho r^2 \textstyle\sum_{m\neq k}^{\infty} &\left[(z-a_m)^{-2}\overline{\psi_m\left(a_m + \frac{r^2}{z-a_m}\right)} - a_m^{-2}\,\overline{\psi_m(a_m)}\right] \\ &-\varrho r^2\overline{\psi_k(a_k)} + q(r), \quad |z-a_k| \le r \ (k = 1,2,\ldots). \end{aligned} \tag{15}$$

where [14, formula (4.7)]

$$q(r) = 1 + \varrho r^2 \lim_{\gamma\ni\Delta z\to\infty}\sum_{k=1}^{\infty}\left[\frac{1}{a_k^2} - \frac{1}{(w-a_k)(w-a_k-\Delta z)}\right] + O(r^4). \tag{16}$$

Here, the limit is considered along a simple curve $\gamma \subset D^-$ connecting a point $w \in D^-$ and infinity. Without loss of generality, one may take $w = 0$. An algorithm to construct the higher order terms of $q(r)$ is described in [14]. The relation (16) becomes

$$q(r) = 1 + \varrho r^2 S_2' + O(r^4), \tag{17}$$

where

$$S_2' = \lim_{\gamma\ni\Delta z\to\infty}\sum_{k=1}^{\infty}\left[\frac{1}{a_k^2} - \frac{1}{a_k(a_k+\Delta z)}\right]. \tag{18}$$

It is assumed that the limit (18) does not depend on the choice of the curve γ. The infinite sum in (18) goes in accordance with the Eisenstein summation defined by (10) and (14). In the case of a double periodic set $\mathcal{A}$, the value S_2' becomes S_2.

The convergence of series in (15) can be justified through the Taylor expansion

$$\psi_m(z) = \sum_{l=0}^{\infty} \alpha_l (z-a_m)^l, \quad |z-a_m| < r, \tag{19}$$

and the inversion

$$\overline{\psi_m\left(a_m + \frac{r^2}{\overline{z-a_m}}\right)} = \sum_{l=0}^{\infty} \overline{\alpha_l} \left(\frac{r^2}{z-a_m}\right)^l, \quad |z-a_m| > r. \tag{20}$$

Substitution of (20) into (15) yields the functional equation

$$\psi_k(z) = \varrho \sum_{l=1}^{\infty} \overline{\alpha_l} r^{2(l+1)} \sum_{m\neq k}^{\infty} \left(\frac{1}{z-a_m}\right)^{l+2} + \varrho r^2 \overline{\psi_k(a_k)} + 1, \quad |z-a_k| \le r. \tag{21}$$

Here, the absolute, almost uniform convergence in $z \notin \mathcal{A}$ follows from the absolute, almost uniform convergence of the series [14]

$$F_p(z) := \sum_{m=1}^{\infty} \frac{1}{(z-a_m)^p}, \quad p \ge 3. \tag{22}$$

3. Low order formula for the effective conductivity

Introduce the function $\Psi(z) = \psi_k(z)$ in $|z-a_k| \le r$ for all $k = 1, 2, \ldots$. The function $\Psi(z)$ is analytic in the non-connected domain D^+ and continuous in its closure. Such functions form the Banach space $\mathcal{C}(D^+)$ with the norm $\|\Psi\| = \sup_{z\in D^+\cup\partial D^+} |\Psi(z)|$. The system of functional equations (15) can be written as an equation in the space $\mathcal{C}(D^+)$

$$\Psi = \varrho r^2 B\Psi + 1 \tag{23}$$

where the operator B is defined by the right part of (15). It was proved in [14] that the operator B is bounded, and a method of successive approximations can be applied to (23) in the space $\mathcal{C}(D^+)$. It is worth noting that equation (23) can be considered in the Hardy-type space following [6].

Applying iterations to (23) and substituting $r^2 = \frac{f}{\pi}$ we obtain

$$\psi_k(z) = 1 + \frac{\varrho f}{\pi} \left\{ \sum_{m\neq k}^{\infty} \left[(z-a_m)^{-2} - a_m^{-2}\right] + S_2' \right\} + O(f^2), \quad \text{as } f \to 0. \tag{24}$$

The components of the effective conductivity tensor can be calculated by the formula [14]

$$\lambda_{xx} - \mathrm{i}\lambda_{xy} = 1 + 2\varrho f \ \langle\psi\rangle. \tag{25}$$

where

$$\langle\psi\rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi_k(a_k). \tag{26}$$

Substitution of (24) into (26) yields

$$\lambda_{xx} - \mathrm{i}\lambda_{xy} = 1 + 2\varrho f + 2\varrho^2 f^2 \frac{1}{\pi} \left\{ S_2' + \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{m\neq k}^{\infty} \left[\frac{1}{(a_k-a_m)^2} - \frac{1}{a_m^2}\right] \right\} + O(f^3). \tag{27}$$

Following [19] one can consider the problem for the structure $\mathcal{A}^*$ obtained from $\mathcal{A}$ by rotation about the angle $\frac{\pi}{2}$, i.e., $\mathcal{A}^* = \{\mathrm{i}a_1, \mathrm{i}a_2, \ldots\}$. Then,

$$\lambda_{yy} + \mathrm{i}\lambda_{xy} = 1 + 2\varrho f + 2\varrho^2 f^2 \frac{1}{\pi} \left\{ {S'}_2^* - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{m\neq k}^{\infty} \left[\frac{1}{(a_k-a_m)^2} - \frac{1}{a_m^2}\right] \right\} + O(f^3). \tag{28}$$

Calculate the invariant

$$\frac{1}{2}(\lambda_{xx}+\lambda_{yy}) = 1+2\varrho f+2\varrho^2 f^2 \frac{1}{2\pi}(S_2'+{S'}_2^*)+O(f^3). \tag{29}$$

Here, we use the absolute convergence of the double series $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{m\neq k}^{\infty}$. The series S_2' converges conditionally; hence, we cannot directly replace a_k by $\mathrm{i}a_k$. However, to compute ${S'}_2^*$ we can apply the formula [19]

$$S_2'+{S'}_2^* = 2\pi, \tag{30}$$

and get the famous Clasius-Mossotti (Maxwell) approximation

$$\frac{1}{2}(\lambda_{xx}+\lambda_{yy}) = 1+2\varrho f+2\varrho^2 f^2+O(f^3) = \frac{1+\varrho f}{1-\varrho f}+O(f^3). \tag{31}$$

Consider the regular square array $\mathcal{A}=\{m_1+\mathrm{i}m_2:\ m_1,m_2\in\mathbb{Z}\}$. This example was investigated by Rayleigh [18] whi proved that $S_2=\pi$. Calculate the following limit using the Eisenstein summation

$$\begin{aligned}
&\alpha = \lim_{M_1\to\infty}\lim_{M_2\to\infty}\sum_{k_2=-M_2}^{M_2}\sum_{k_1=-M_1}^{M_1}\frac{1}{(2M_1+1)(2M_2+1)}\\
&\times\sum\nolimits'^{(e)}_{(m_1,m_2)\neq(k_1,k_2)}\left[\frac{1}{(k_1+\mathrm{i}k_2-m_1-\mathrm{i}m_2)^2}-\frac{1}{(m_1+\mathrm{i}m_2)^2}\right]=\\
&\lim_{M_1\to\infty}\lim_{M_2\to\infty}\sum_{k_2=-M_2}^{M_2}\sum_{k_1=-M_1}^{M_1}\frac{1}{(2M_1+1)(2M_2+1)}\\
&\times\sum\nolimits'^{(e)}_{(m_1,m_2)\neq(k_1,k_2)}\frac{1}{(k_1+\mathrm{i}k_2-m_1-\mathrm{i}m_2)^2}-\sum\nolimits'^{(e)}_{m_1,m_2}\frac{1}{(m_1+\mathrm{i}m_2)^2}.
\end{aligned} \tag{32}$$

Here, the prime means that the term $(m_1,m_2)=(0,0)$ is skipped; $n=(2M_1+1)(2M_2+1)$. The first sum in the fourth line does not depend on $k_1+\mathrm{i}k_2$. Therefore, α vanishes, and (27) for the square array becomes

$$\lambda_{xx}-\mathrm{i}\lambda_{xy} = 1+2\varrho f+2\varrho^2 f^2+O(f^3). \tag{33}$$

Taking into account the macroscopic isotropy of the square array, we get

$$\lambda_{xx}=\lambda_{yy}=1+2\varrho f+2\varrho^2 f^2+O(f^3),\quad \lambda_{xy}=0. \tag{34}$$

An exact formula for λ_{xx} was derived in [16]. Its first terms coincide with (34).

4. Conclusion

The general assumption of strictly stationary fields in the theory of random composites is not constructive. This is the reason why it is not checked in practice during the derivation of formulas for the effective constants. This neglect of the homogenization principles may lead to wrong corrections. For instance, as it follows from (31) the Clasius-Mossotti approximation holds for 2D composites up to $O(f^3)$. Analogous formulas hold for 2D elastic fields and 3D problems [17, Chapter 9].

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Комплекс жазықтықтағы құрылымдық сомалар және оларды композициялық материалдарға қолдану

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Аңдатпа. Бір жақтауға салынған бірдей дөңгелек дискілерден тұратын екі өлшемді екі компонентті композитті қарастырайық. Қарастырылып отырған композиттердің өткізгіштігі олардың компоненттері алып жатқан аудандарда аналитикалық және олардың тұйықталуында үзіліссіз функциялардың сызықтық түйіндесінің математикалық есептері арқылы модельденеді. Бұл дисперсті композиттердің тиімді электр өткізгіштігін қатаң стационарлық өрістерді қамтитын конструктивті гомогенизация теориясы арқылы анықтауға болады. Эйзенштейннің жинақтау әдісі гомогенизация процесінде пайда болатын шартты жинақталатын қосындыларды зерттеуде қолданылады. Сонымен қатар, $O(f^3)$ дәлдікті макроскопиялық изотропты екі өлшемді композиттерге қатысты Клаузиус-Моссоттидің жуықтауы кездейсоқ композиттерге негізделген. Макроскопиялық анизотропты композиттердің тиімді өткізгіштік тензоры үшін $O(f^3)$ дәлдігімен жаңа аналитикалық формула алынды. Бұл формула Эйзенштейннің қосындылау әдісі арқылы есептелген дискілердің квадрат массиві үшін Рэлей торының қосындысымен беттесетін шартты жинақты S_2' қосындысын қамтиды. Сонымен қатар, S_2' ақырлы нүкте мен шексіздікті қосатын қисыққа тәуелді. Бұл тәсіл осындай материалдардың өзгеруін түсінуге сенімді негіз беріп, олардың электр өткізгіштігінің тиімді қасиеттерін сипаттайды.

Ключевые слова: құрылымдық сомалар, композиттердің тиімді қасиеттері, гомогенизация, қатаң стационарлық өріс, R - сызықтық есеп, асимптотикалық талдау.

Структурные суммы на комплексной плоскости и их применения к композиционным материалам

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Аннотация. Рассматривается двумерный двухкомпонентный композит, состоящий из непересекающихся идентичных круглых дисков, вставленных в однородную среду. Проводимость рассматриваемых композитов моделируется с помощью математической задачи линейного сопряжения для функций, аналитических в областях, занимаемых компонентами композитов, и непрерывных в их замыканиях. Эффективная электропроводность этих дисперсных композитов может быть определена с помощью конструктивной теории гомогенизации, которая предполагает строго стационарные поля. Метод суммирования Эйзенштейна применяется для анализа условно сходящихся сумм, возникающих в процессе гомогенизации. Кроме того, приближение Клаузиуса-Моссотти, которое справедливо для макроскопически изотропных двумерных композитов с точностью до $O(f^3)$, обосновано для случайных композитов. Получена новая аналитическая формула для тензора эффективной проводимости макроскопически анизотропных композитов с точностью до $O(f^3)$. Эта формула содержит условно сходящуюся сумму S_2', совпадающую с суммой решетки Рэлея, вычисленной методом суммирования Эйзенштейна. Более того, S_2' зависит от кривой, соединяющей конечную точку с бесконечностью. Этот подход обеспечивает теоретическую основу для понимания поведения таких материалов и позволяет получить представление об их эффективных свойствах электропроводности.

Ключевые слова: структурная сумма, эффективные свойства композитов, гомогенизация, строго стационарное поле, R-линейная задача, асимптотический анализ.

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