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DECAY OF THE INITIAL OIL CONCENTRATION DISCONTINUITY IN THE BUCKLEY–LEVERETT MODEL [1]

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Abstract. We consider a free boundary problem for a one-dimensional system of Buckley-Leverett equations, describing the displacement of oil by a suspension. For this problem we formulated conditions for the strong decay of the discontinuity of the initial oil concentration. We will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity in the initial concentration of oil, when at rest in one half of the domain there is oil, and in the other half of the domain there is a suspension, and these domains are separated by an impenetrable partition. At the initial moment of time, the partition is removed and a non-negative suspension velocity is maintained at the injection wells. An accurate analysis of the unique solution to the Buckley-Leverett model shows that at the initial moment of time, oil begins to displace the suspension, resulting in the formation of a zone of mixing of oil and suspension. If the velocity of the suspension at the injection wells is high enough, then at some point in time the natural option of displacing oil by the suspension begins to be realized.

Keywords: Free boundary problems, transport equations, displacement of oil by suspension, strong discontinuity conditions.

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1. Introduction and main results

Free boundary problems for differential equations are some of the most difficult in the theory of partial differential equations. In these problems, along with solving differential equations, it is necessary to determine the domain in which this solution is sought. As a rule, this domain (boundary) is determined from an additional boundary condition at the free boundary. In the theory of free boundary problems, the Stefan problem, the Masket problem, and the Heele-Shaw problem [1]–[5] for the heat or Laplace equations are well known. These problems are formulated quite simply, but so far the existence of a classical solution has been proven only locallty in time (excluding some simple cases). As for systems of differential equations, here we should note the works of V. A. Solonnikov for free boundary problems to the Navier-Stokes system [6], [7] and A. Friedman [8].

But, as in the case of one equation, here it is possible to prove only local existence of a classical solution [6], [7], or limit oneself to a phenomenological model that describes the physical process at the macroscopic level [8].

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Separately, there is a large class of free boundary problems for the equations of gas dynamics and hydrodynamics of an ideal incompressible fluid. These problems are well studied and have a rich history [9]-[11].

We will consider the Buckley–Leverett model, formulated in [12] and describing the displacement of oil by a suspension in the pore space of the absolutely rigid solid skeleton at the macroscopic level. Let us recall that such models are called ***Phenomenological***.

The existence and uniqueness of a generalized solution to the system of Buckley–Leverett equations for smooth data of the problem was proved by S. N. Antontsev and V. N. Monakhov [13]. We will be interested in the structure of the weak solution of the Buckley–Leverett system of equations for a discontinuous initial oil concentration. In the terminology of L.V. Ovsyannikov (Appendix A in [14]), such a problem is called ***Problem on the decay of strong discontinuity***.

For simplicity of presentation, we restrict ourselves to the case of one spatial variable.

We look for the solution to the Buckley–Leverett system in the domain $\Omega_T = \Omega \times (0,T)$, $\Omega = (0,1) \subset \mathbb{R} = (-\infty, \infty)$, consisting of Darcy's system of filtration

$$v_{ol} = -\frac{k}{\mu_{ol}} f_{ol}(c) \frac{\partial p_{ol}}{\partial x}, \tag{1}$$

$$v_{sp} = -\frac{k}{\mu_{sp}} f_{sp}(c) \frac{\partial p_{sp}}{\partial x} \tag{2}$$

and laws of conservation of mass

$$\frac{\partial}{\partial t}(mc) + \frac{\partial v_{ol}}{\partial x} = 0, \tag{3}$$

$$\frac{\partial}{\partial t} m(1-c) + \frac{\partial v_{sp}}{\partial x} = 0. \tag{4}$$

The system (1) – (4) is completed with the state equations

$$k\, p_{ol} - k\, p_{sp} = \alpha_{cap}\, c, \tag{5}$$

$$f_{ol}(c) = \alpha_{ol}\, c, \;\; f_{sp}(c) = \alpha_{sp}\,(1-c) \tag{6}$$

and following boundary and initial conditions

$$v_{sp}(0,t) = 0, \;\; v_{ol}(1,t) = 0, \tag{7}$$

$$c(x,0) = c^0(x). \tag{8}$$

In (1) – (8) c is a concentration of oil in the pore liquid, $(1-c)$ is a concentration of suspension in the pore liquid, v_o is the oil velocity, v_{sp} is the suspension velocity, p_{ol} is the oil pressure, p_{sp} is the suspension pressure, μ_{ol} is the dimensionless oil viscosity and μ_{sp} is the dimensionless suspension viscosity.

The positive constants α_{ol}, α_{sp} and α_{cap} are supposed to be known.

First of all, we transform the equations (1) - (6) into a convenient for us form:

$$\frac{f_{ol}}{\mu_{ol}} k \frac{\partial p_{ol}}{\partial x} + \frac{f_{sp}}{\mu_{sp}} k \frac{\partial p_{sp}}{\partial x} = 0, \tag{9}$$

$$(\frac{f_{ol}}{\mu_{ol}}(k \frac{\partial p_{sp}}{\partial x} + \alpha_{cap} \frac{\partial c}{\partial x}) + \frac{f_{sp}}{\mu_s} k \frac{\partial p_{sp}}{\partial x} = 0, \tag{10}$$

$$k \frac{\partial p_{ol}}{\partial x} = \frac{\alpha_{cap} \mu_{ol}\, f_{sp}}{(\mu_{ol}\, f_{sp} + \mu_{sp}\, f_{ol})} \frac{\partial c}{\partial x}, \tag{11}$$

$$k \frac{\partial p_{sp}}{\partial x} = -\frac{\alpha_{cap} \mu_{sp}\, f_{ol}}{(\mu_{ol}\, f_{sp} + \mu_{sp}\, f_{ol})} \frac{\partial c}{\partial x}, \tag{12}$$

$$v_{ol} = -\frac{f_{ol}}{\mu_{ol}} k \frac{\partial p_{ol}}{\partial x} = -\frac{\alpha_{cap}\, f_{ol}\, f_{sp}}{(\mu_{ol}\, f_{sp} + \mu_{sp}\, f_{ol})} \frac{\partial c}{\partial x} = m\, c\, u_{ol},$$

$$u_{ol} = -\varphi_{ol}(c) \frac{\partial c}{\partial x}, \tag{13}$$

$$v_{sp} = -\frac{fsp}{\mu_{sp}}\, k\, \frac{\partial p_{sp}}{\partial x} = \frac{\alpha_{cap}\, f_{ol}\, f_{sp}}{(\mu_{ol}\, f_{sp} + \mu_{sp}\, f_{ol})}\frac{\partial c}{\partial x} = m\,(1-c)u_{sp},$$

$$u_{sp} = \varphi_{sp}(c)\frac{\partial c}{\partial x}, \tag{14}$$

$$\varphi_{ol}(c) = \alpha_{\varphi}\frac{(1-c)}{(a-bc)},\;\; \varphi_{sp}(c) = \alpha_{\varphi}\frac{c}{(a-bc)},$$

$$\varphi(c) = \alpha_{\varphi}\frac{c\,(1-c)}{(a-bc)} = c = (1-c)\varphi_{sp}(c) = c\varphi_{ol}(c), \tag{15}$$

$$\frac{\partial}{\partial t}c + \frac{\partial}{\partial x}(cu_{ol}) = 0,\;\; \frac{\partial}{\partial t}(1-c) + \frac{\partial}{\partial x}\big((1-c)u_{sp}\big) = 0, \tag{16}$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\big(\varphi(c)\frac{\partial c}{\partial x}\big). \tag{17}$$

In (9) – (17)

$$\alpha_{\varphi} = m^{-1}\,\alpha_{ol}\,\alpha_{sp}\,\alpha_{cap},\;\; a = \mu_{ol}\,\alpha_{sp} + \mu_{sp}\,\alpha_{ol},\;\; b = \mu_{ol}\,\alpha_{sp} < a. \tag{18}$$

The Buckley–Leverett model and its analogs (see [15] - [17] and literature cited there) are phenomenological mathematical models and serve as the basis model for existing hydrodynamic simulators, such as ***Eclipse***, ***Black Oil*** (Schlumberger), ***Tempest*** (Roxar), ***VIP*** (Landmark) and ***TimeZYX*** (Standard Oil and Trust).

Note that a hydrodynamic simulator is a certain ***Scale*** (set) of mathematical models of an oil reservoir of varying degrees of accuracy, supplemented with digital characteristics of physical properties, such as density and elastic properties of the solid skeleton, soil, density and viscosity of filtered liquids, as well as geometric characteristics of the reservoir in consideration, such as the structure of the solid skeleton, the geometry of the domain occupied by the reservoir, and visualization programs for numerical implementations.

Existing simulators, according to their purpose, must adequately reflect the simulated physical process.

Do existing simulators solve this problem?

Let's reformulate the question differently.

Since the basis of any hydrodynamic simulator is the corresponding scale of mathematical models (ideally!), the question can be formulated as follows:

Do the existing mathematical models underlying existing hydrodynamic simulators adequately reflect the simulated physical process?

Only adequate mathematical models will be able to optimize the oil production process, and only with adequate modeling can the main task of a hydrodynamic simulator be solved - this, of course, is ***Maximum benefit*** during the exploitation of a field.

A positive or negative answer depends on what exactly ***Adequacy of a mathematical model for a given physical process means.***

To do this, it is necessary to formulate ***Adequacy Criteria*** of the mathematical model.

In the case of phenomenological models, the criterion of adequacy can only be experiment. Is experiment a criterion of adequacy?

The answer is NO.

In fact, it makes no sense to talk about an experiment, since in any phenomenological model there are enough free parameters and even functions that are in no way related to the geometry of the reservoir (porosity and structure of the pore space) or to the physical characteristics of the displacement process (viscosity and density of filtered liquids and density and elastic properties of the solid skeleton). Therefore, by varying the indicated constants and functions, one can achieve agreement with any experiment!

Let us recall that by its definition, any phenomenological mathematical model is a set of postulates (axioms), expressed using differential equations, supplemented by corresponding boundary and initial conditions, and defining relations (equalities). In this case, the characteristic dimensions in macroscopic models are meters or tens of meters. Because of this, these models do not distinguish either the microstructure of a continuous medium, or the free boundary separating liquids, or the peculiarities of the interaction of fluids with the solid skeleton of the soil (adhesion or sliding conditions), since in such a model at each point of the continuous medium there is both rock (hard skeleton) together with the liquid in the pores of this skeleton, and the free boundary separating the various components of the medium. All such models are built on the same principle. Fluid dynamics, as a rule, are controlled by the Darcy filtration equation system or some modification thereof, and the interaction of fluids is governed by the laws of mass conservation for each fluid. But all fundamentally important changes occur precisely at the microscopic level, corresponding to the average size of pores or cracks in rocks, while any of the proposed macroscopic models operates on completely different (orders of magnitude larger) scales, which explains their diversity. The authors of such models simply do not have an accurate method for describing physical processes at the microscopic level based on the fundamental laws of Newtonian continuum mechanics, nor the ability to take into account the microstructure of rocks in macroscopic models. Therefore, they have to limit themselves to certain speculative considerations (postulates) formulated by the authors themselves.

In view of the above, a natural question arises: if there are several macroscopic models describing the same physical process, which of them most adequately reflects this process? Where is the criterion of truth here?

The answer to this question is quite complex and is beyond the scope of this article. Let's just say that in order to derive a macroscopic model adequate to the physical process under consideration, it is first necessary, following the principles formulated in the works of J. B. Keller [18] and E. Sanchez-Palencia [19], to describe this process based on the equations of Newton's classical mechanics at the microscopic level (average size of tens of microns) and only then, using mathematically strict averaging (homogenization), derive a macroscopic model that most accurately describes this physical process.

In this publication we will prove that the phenomenological Buckley-Leverett model does not adequately describe the physical process under consideration. To do this, we will study the problem of the decay of a discontinuity of the initial concentration of oil, when at rest in one half of the domain there is oil, and in the other half of the domain there is a suspension, and these domains are separated by an impenetrable partition. At the initial moment of time, the partition is removed and at the injection wells a non-negative suspension velocity is maintained. An accurate analysis of the unique solution to the Buckley-Leverett model shows that at the initial moment of time, oil begins to displace the suspension, resulting in the formation of a zone of mixing of oil and suspension. If the velocity of the suspension at injection wells is high enough, then at some point in time the natural option of displacing oil by suspension begins to take place.

Everywhere below we use the notation of functional spaces and norms in these spaces adopted in [20] and [21].

2. Auxiliary statements

2.1. Domain and boundaries

As $\Omega_{ol}(t)=\{x\in\Omega: 0<R_{ol}(t)<x<1\}$ we denote the domain occupied by oil, as $\Omega_{m}(t)=\{x\in\Omega: R_{sp}(t)<x<R_{ol}(t)<x<1\}$ –the domain occupied by the mixture of oil and suspension and as $\Omega_{sp}(t)=\{x\in\Omega: 0<x<R_{sp}\}$ we denote the domain occupied by the suspension.

Here $R_{sp}(t)$ is the boundary between $\Omega_{sp}(t)$ and $\Omega_{m}(t)$ and $R_{ol}(t)$ is the boundary between $\Omega_{m}(t)$ and $\Omega_{ol}(t)$.

Let also $\Omega_{ol,T}=\bigcup\limits_{t=0}^{t=T}\Omega_{ol}(t)$, $\Omega_{m,T}=\bigcup\limits_{t=0}^{t=T}\Omega_{m}(t)$, $\Omega_{sp,T}=\bigcup\limits_{t=0}^{t=T}\Omega_{sp}(t)$,

$\Gamma_{ol}(t)=\{x\in\Omega: x=R_{ol}(t)\}$, $\Gamma_{sp}(t)=\{x\in\Omega: x=R_{sp}(t)\}$,

$\Gamma_{T,sp}=\bigcup\limits_{t=0}^{t=T}\Gamma_{sp}(t)$ and $\Gamma_{T,ol}=\bigcup\limits_{t=0}^{t=T}\Gamma_{sol}(t)$.

2.2. Derivation of boundary conditions on strong discontinuities

Suppose that during the diffusion of oil and suspension between domain $\Omega_{ol,T}$, occupied by oil and domain $\Omega_{sp,T}$, occupied by suspension, instantly formed domain $\Omega_{m,T}$, occupied by mixture of oil and suspension.

According supposition $c = 1$ in $\Omega_{ol,T}$ and $c = 0$ in $\Omega_{sp,T}$.

Next we will derive boundary conditions on strong discontinuities $\Gamma_{sp,T}$ and $\Gamma_{ol,T}$ following [14].

Recall, that for the case of one spacial variable for equation

$$\frac{\partial \widetilde{F}}{\partial t} + \frac{\partial}{\partial x}(\widetilde{F}\,\widetilde{u}) = 0 \tag{19}$$

(equality (A.6.4) in [5]) where $\widetilde{F} = F_{sp}$ as $(x,t) \in \Omega_{sp,T}$, $\widetilde{F} = F$ as $(x,t) \in \Omega_{m,T}$ and $\widetilde{F} = F_{ol}$ as $(x,t) \in \Omega_{ol,T}$, $\widetilde{v} = v_{sp}$ s $(x,t) \in \Omega_{sp,T}$, $\widetilde{v} = v$ as $(x,t) \in \Omega_{m,T}$ и $\widetilde{v} = v_{ol}$ as $(x,t) \in \Omega_{ol,T}$, the jump of functions $\widetilde{F}$ and $\widetilde{v}$ at the strong discontinuities $\Gamma_{sp,T} = \{x \in \Omega = (0,1) : x = R_{sp}(t)\}$ are defined from relations

$$[\widetilde{F}(\frac{dR_{sp}}{dt} - \widetilde{u})]_{\Gamma_{sp}} = F_{sp}(R_{sp}(t),t)\big(\frac{dR_{sp}}{dt}(t) - u_{sp}(R_{sp}(t),t)\big) - \\ - F(R_{sp}(t),t)(\frac{dR_{sp}}{dt}(t) - u(R_{sp}(t),t)) = 0 \tag{20}$$

(equality (A6.12) in [14]).

In the same way we get the equality

$$[\widetilde{F}(\frac{dR_{ol}}{dt} - \widetilde{u})]_{\Gamma_{ol}} = F(R_{ol}(t),t)(\frac{dR_{ol}}{dt}(t) - u(R_{ol}(t),t)) - \\ - F_{ol}(R_{ol}(t),t)\big(\frac{dR_{ol}}{dt}(t) - u_{ol}(R_{sp}(t),t)\big) = 0 \tag{21}$$

on the boundary $\Gamma_{ol}(t)$.

In section 4 we prove that $c = 0$ in $\overline{\Omega}_{sp}(t)$, $c = 1$ in $\overline{\Omega}_{ol}(t)$ and

$$c(x,t) > 0 \text{ as } R_{sp}(t) \leqslant x \leqslant R_{ol}(t) \text{ and } 0 < t < T. \tag{22}$$

Thus, for equations (21) in the form

$$\frac{\partial \widetilde{c}}{\partial t} + \frac{\partial}{\partial x}(\widetilde{c}\,\widetilde{u}) = 0, \tag{23}$$

where

$$\widetilde{c} = \begin{cases} 0 & \text{in } \Omega_{sp}(t), \\ c & \text{in } \Omega_m(t), \end{cases}$$

$$\widetilde{u} = \begin{cases} 0 & \text{in } \Omega_{sp}(t), \\ u_{ol} = -u_{sp} & \text{in } \Omega_m(t), \end{cases}$$

and on the boundary $\Gamma_{sp}(t)$ holds true equality

$$[\widetilde{c}(\frac{dR_{sp}}{dt} - \widetilde{u})] = c(R_{sp}(t),t)(\frac{dR_{sp}}{dt} - u_{ol}) = 0, \tag{24}$$

which implies

$$\frac{dR_{sp}}{dt} = u_{ol}(R_{sp}(t),t) < 0. \tag{25}$$

In the same way for equation (23), where

$$\widetilde{c} = \begin{cases} c & \text{in } \Omega_m(t), \\ 1 & \text{in } \Omega_{ol}(t), \end{cases}$$

$$\widetilde{u} = \begin{cases} u_{ol} = -u_{sp} & \text{in } \Omega_m(t), \\ 0 & \text{in } \Omega_{ol}(t) \end{cases}$$

we get

$$c(R_{ol}(t),t)(\frac{dR_{ol}}{dt}-u_{sp})=0, \tag{26}$$

and, consequently,

$$\frac{dR_{ol}}{dt}=u_{sp}(R_{ol}(t),t)>0. \tag{27}$$

2.3. Spaces $\mathbb{BV}(\Omega)$ and $\mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ of functions of bounded variation. Helly's selection principle

In the present publication we restrict ourself with spaces $\mathbb{BV}(\Omega)$ and $\mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ of functions of bounded variation in $\Omega\subset\mathbb{R}$ [22].

Definition 1. We call the closure of all infinitely smooth functions $u(x)$ in the norm

$$\|u\|_{BV(\Omega)}=\big(\int_\Omega(|u(x)|+|\frac{du}{dx}|)dxdt\big) \tag{28}$$

as the space of functions of bounded variation $\mathbb{BV}(\Omega)$.

Definition 2. The closure of all infinitely smooth functions $u(x,t)$ in the norm

$$\|u\|_{BV(\Omega_T)}=\max_{0\leqslant t\leqslant T}\big(\int_\Omega(|u(x,t)|+|\frac{\partial u}{\partial x}|)dxdt\big) \tag{29}$$

is called the space of functions of bounded variation $\mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$.

Theorem 1. [22]

1) A function $u(x)$ belongs to the space $\mathbb{BV}(\Omega)\big)$ if and only if there exists some constant $K>0$ such that

$$\int_\Omega|u(x+h)-u(x)|dx\ \leqslant\ K\,|h| \tag{30}$$

for all $h\in\mathbb{R}$.

2)A function $u(x,t)$ belongs to the space $\mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ if and only if there exists some constant $K>0$ such that

$$\max_{0\leqslant t\leqslant T}\int_\Omega|u(x+h,t)-u(x,t)|dx\ \leqslant\ K\,|h| \tag{31}$$

for all $h\in\mathbb{R}$.

Let

$$c_n^0(x)=\begin{cases}0 & \text{for } 0\ \leqslant\ x\ \leqslant\ \frac{1}{2}-\frac{1}{n},\\ \frac{n}{2}\,(x-\frac{1}{2}+\frac{1}{n}) & \text{for } \frac{1}{2}-\frac{1}{n}<x\ \leqslant\ \frac{1}{2}+\frac{1}{n},\\ 1 & \text{for } x>\frac{1}{2}+\frac{1}{n},\end{cases} \tag{32}$$

Lemma 1. *The sequence $\{c_n^0\}$ is monotone increasing sequence of monotone increasing functions $c_0^n\in\mathbb{BV}(\Omega)$ and*

$$\|c_n^0-c^0\|_{\mathbb{BV}}\to 0\ \text{ as }\ n\to\infty. \tag{33}$$

The proof of statements follows from the definition of functions c_n^0 and Theorem 1.

Definition 3. We say that the function $u(x,t)$, bounded in $\mathbb{L}_2(\Omega_T)$, possesses the time derivative $\dfrac{\partial u}{\partial t}\in\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$, if

$$|\int\int_{\Omega_T}u\,\frac{\partial\xi}{\partial t}dxdt|\leqslant M_u\,|\int\int_{\Omega_T}|\nabla\xi|^2dxdt|^{\frac{1}{2}}$$

for all functions $\xi\in\mathbb{W}_2^{1,1}(\Omega_T)$ with some positive constant M_u independent of ξ.

Remark 1. We denote the norm of an element φ in $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$ as $\|\varphi\|_{W_2^{-1}}$.

Lemma 2. *(**Helly's selection principle**.) Let sequences $\{u_n\}$ is bounded in the space $\mathbb{BV}(\Omega)$,and the sequence itself is bounded in $\mathbb{L}_\infty(\Omega)$:*

$$|u_n| \leqslant K, \ \ \|u\|_{BV(\Omega)} \leqslant K. \tag{34}$$

Then there exists some subsequence of the sequence $\{u_n\}$ convergent almost everywhere in $\mathbb{L}_2(\Omega)$. [23].

Here and below, K will denote constants independent of N .

Consequence 1. Let the sequence $\{u_n\}$ converges almost everywhere in Ω and $0 \leqslant u_n \leqslant 1$. Then it converges in $\mathbb{L}_2(\Omega)$.

Lemma 3. *Let sequences $\{u_n\}$ is bounded in the space $\mathbb{BV}(\Omega)$ and the sequence of derivatives $\{\dfrac{\partial u_n}{\partial t}\}$ is bounded in the space $\mathbb{L}_2\big(0,T;\mathbb{W}_2^{-1}(\Omega)\big)$.*

Then there exists some subsequence of the sequence $\{u_n\}$ strongly convergent in $\mathbb{L}_2(\Omega_T)$.

The proof of this lemma repeats the proof of the compactness lemma in Lions [21].

3. Main result

Definition 4. A function $c \in \mathbb{BV}(\Omega_T)$ is called a weak solution to the problem (1) – (8) if

$$\int_\Omega c(x,t_0)\,\xi(x,t_0)dx + \int_0^{t_0}\int_\Omega \big(-m\,c\frac{\partial \xi}{\partial t} + \varphi(c)\frac{\partial c}{\partial x}\frac{\partial \xi}{\partial x}\big)dxdt =$$
$$= \int_\Omega c^0(x)\xi(x,0)dx + \int_0^{t_0}\big(\xi(1,t)u_1(t) - \xi(0,t)u_0(t)\big)\big)dt \tag{35}$$

for all infinitely smooth functions $\xi(x,t)$ in Ω_T .

Theorem 2. *The problem* (1) – (8) *has a unique weak solution.*

4. Proof of Theorem 2

4.1. Construction of approximate solutions

Let $c_n(x,t)$ be solution to the approximate diffusion equation

$$\frac{\partial c_n}{\partial t} = \frac{\partial}{\partial x}\Big(\big(\varphi(c_n) + \frac{1}{n}\big)\frac{\partial c_n}{\partial x}\Big), \tag{36}$$

satisfying initial condition

$$c_n(x,0) = c_n^0(x) \tag{37}$$

and boundary conditions

$$\big(\varphi\big(c_n(j,t) + \frac{1}{n}\big)\big)\frac{\partial c_n}{\partial x}(j,t) = 0, \ j = 0,\, 1. \tag{38}$$

The problem (36)–(38) has an unique monotone increasing classical solution $c_n \in \mathbb{C}^{2,1}(\Omega_T) \cap \mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ for all $n > 0$.

The existence of such solution follows from [1], and its monotonicity follows from maximum principle.

Indeed, the following statement holds true

Lemma 4. *For all $n > 0$ $\dfrac{\partial c_n}{\partial x}(x,t) \geqslant \alpha > 0$ in Ω_T , where $\alpha = const > 0$.*

Proof. Note, that in accordance with [1] functions c_n are infinitely smooth in variables x and t in the domain $\overline{\Omega}$ for $t > 0$ and satisfy boundary conditions in a usual sense.

Consequently, the nonlinear heat equation (36) can be differentiated with respect to all variables the required number of times and integration by parts can be used.

First of all, we use the obvious maximum principle

$$0 \leqslant c_n \leqslant 1. \tag{39}$$

Next we define new functions $w_n = \dfrac{\partial c_n}{\partial x}$.

The direct differentiation of (36) in variable x gives us

$$\frac{\partial w_n}{\partial t} = \Big(\big(\varphi(c_n) + \frac{1}{n}\big)\frac{\partial^2 w_n}{\partial x^2} + 3\,\varphi'(c_n)\, w_n \frac{\partial w_n}{\partial x} + \varphi''(c^n)\,(w_n)^3, \tag{40}$$

$$\Big(\varphi\big(c(j,t)\big) + \frac{1}{n}\Big) w_n(j,t) = 0, \;\; j = 0,\, 1, \tag{41}$$

$$w_n(x,0) = \frac{\partial c_n^0}{\partial x}(x) \geqslant 0. \tag{42}$$

Let's show that $\dfrac{d^2\varphi_n}{dy^2}(y) \leqslant 0$ for $0 \leqslant y \leqslant 1$, that, taking into account the boundary conditions (41) and the strict maximum principle [1], immediately guarantees us the required result.

Turning to the equalities (15) we obtain for the function $\psi(z \equiv \varphi(c)$, where $z = a - b\,c$, following relations:

$$\varphi''(c) = b^2\,\psi''(z),\;\; \psi'(z) = \frac{\alpha_c\, a\, d}{z^2} - 1,\;\; \psi''(z) = -\frac{2\,\alpha_c\, a\, d}{z^3} \leqslant -\frac{2\,\alpha_c\, a}{d^2} < 0,$$

$$\varphi''(c) = b^2\,\psi''(z) \leqslant -\frac{2\,b^2\,\alpha_c\, a}{d^2} = \alpha = const < 0, \tag{43}$$

which completes the proof of the lemma. □

4.2. Limiting procedure as $n \to \infty$

Let

$$\chi_n(x,t) = \int_0^{c_n(x,t)} (\varphi(y) + \frac{1}{n})dy,\;\; \frac{\partial \chi_n}{\partial x} = (\varphi(y) + \frac{1}{n})\frac{\partial c_n}{\partial x}. \tag{44}$$

Then equation (36) takes the form

$$\frac{\partial c_n}{\partial t} = \frac{\partial^2 \chi_n}{\partial x^2}, \tag{45}$$

which we will rewrite as an equivalent integral identity

$$\int_\Omega c_n(x,t_0)\,\xi(x,t_0)dx + \int_0^{t_0}\int_\Omega \big(-m\,c_n\frac{\partial \xi}{\partial t} - \chi_n\frac{\partial^2 \xi}{\partial x^2}\big)dxdt =$$
$$= \int_\Omega c^0(x)\xi(x,0)dx. \tag{46}$$

Lemma 5. *The sequence $\{c_n\}$ contains convergent in $\mathbb{L}_2(\Omega_T)$ subsequence.*

The proof of the lemma follows from Lemma 21.

Renumbering the sequence $\{c_n\}$ we may assume that it converges in $\mathbb{L}_2(\Omega_T)$ to some function $c \in \mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$.

Consequence 2. The sequence $\{\chi_n\}$ converges in $\mathbb{L}_2(\Omega)$ to function $\chi \in \mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ and

$$\chi(x,t) = \int_0^{c(x,t)} \varphi(y)dy,\;\; \frac{\partial \chi}{\partial x} = \varphi(c)\frac{\partial c}{\partial x}. \tag{47}$$

The proof of the statement follows from the definition of χ_n.

Lemma 6. *The sequence $\{c_n\}$ converges in $\mathbb{L}_2(\Omega_T)$ to the weak solution $c \in \mathbb{L}_\infty\big(0,T;\mathbb{BV}(\Omega)\big)$ to the problem* (36)*–*(38)*.*

Proof. To do that we represent the integral identity (35) as

$$\int_{\Omega} c_n(x,t_0)\,\xi(x,t_0)dx - \int_0^{t_0}\int_{\Omega}\big(c_n\,\frac{\partial\xi}{\partial t} + \chi_n\frac{\partial^2\xi}{\partial x^2}\big)dxdt =$$

$$= \int_{\Omega} c_n^0(x)\xi(x,0)dx. \tag{48}$$

Passing to the limit as $n\to\infty$ we arrive at the desired identity

$$\int_{\Omega} c(x,t_0)\,\xi(x,t_0)dx - \int_0^{t_0}\int_{\Omega}\big(c\,\frac{\partial\xi}{\partial t} + \chi\frac{\partial^2\xi}{\partial x^2}\big)dxdt =$$

$$= \int_{\Omega} c(x,t_0)\,\xi(x,t_0)dx - \int_0^{t_0}\int_{\Omega}\big(c\,\frac{\partial\xi}{\partial t} - \frac{\partial\chi}{\partial x}\frac{\partial\xi}{\partial x}\big)dxdt =$$

$$= \int_{\Omega} c(x,t_0)\,\xi(x,t_0)dx - \int_0^{t_0}\int_{\Omega}\big(c\,\frac{\partial\xi}{\partial t} - \varphi(c)\frac{\partial c}{\partial x}\frac{\partial\xi}{\partial x}\big)dxdt =$$

$$= \int_{\Omega} c^0(x)\xi(x,0)dx. \tag{49}$$

□

The uniqueness result proves in a usual way.

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Бакли-Леверетт моделіндегі бастапқы мұнай концентрациясы үзілуінің азаюы

Аннотация. Мұнайды суспензиямен ығыстыруды сипаттайтын Бакли-Леверетт теңдеулерінің бір өлшемді жүйесі үшін еркін шекаралық есептері қарастырылады. Бұл мәселе үшін мұнайдың бастапқы концентрациясының секірістегі қатты үзіліс шарттары тұжырымдалған. Мақалада Бакли-Леверетт феноменологиялық моделі қарастырылып отырған физикалық процесті дұрыс сипаттай алмайтындығы дәлелденді. Ол үшін бір-бірімен араласпайтындай қалқамен бөлінген аймақтың бір жартысында тыныштық күйде мұнай, ал екінші жартысында суспензия болғандағы мұнайдың бастапқы концентрациясындағы үзілістің ыдырау мәселесі зерттеледі. Бастапқы уақытта қалқа жойылып, айдау скважиналарында суспензияның теріс емес жылдамдығы сақталады. Бакли-Леверетт моделінің жалғыз шешіміне жүргізілген нақты анализ бастапқы уақытта мұнай суспензияны ығыстыра отырып, нәтижесінде мұнай мен суспензияның араласу аймағы пайда болады. Егер айдау ұңғымаларында суспензияның қозғалу жылдамдығы жеткілікті жоғары болса, онда белгілі бір уақытта табиғатына сай суспензия керісінше мұнайды ығыстыра бастайды.

Түйін сөздер: Еркін шекаралы есептер, тасымалдау теңдеулері, мұнайды суспензиямен ығыстыру, қатты үзілістегі шарттар.

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Уменьшение начального разрыва концентрации нефти в модели Бакли-Леверетта

Аннотация. Рассматривается задача со свободной границей для одномерной системы уравнений Бакли-Леверетта, описывающей вытеснение нефти суспензией. Для этой задачи сформулированы условия сильного разрыва скачка начальной концентрации нефти. В статье доказано, что феноменологическая модель Бакли-Леверетта неадекватно описывает рассматриваемый физический процесс. Для этого изучается задача о распаде разрыва начальной концентрации нефти, когда в одной половине области покоится нефть, а в другой половине области - суспензия, и эти области разделены непроницаемой перегородкой. В начальный момент времени перегородка удаляется и на нагнетательных скважинах поддерживается неотрицательная скорость суспензии. Точный анализ единственного решения модели Бакли-Леверетта показывает, что в начальный момент времени нефть начинает вытеснять суспензию, в результате чего образуется зона смешивания нефти и суспензии. Если скорость движения суспензии на нагнетательных скважинах достаточно высока, то в какой-то момент времени начинает реализовываться естественный вариант вытеснения нефти суспензией.

Ключевые слова: Задачи со свободными границами, уравнения переноса, вытеснение нефти суспензией, условия на сильном разрыве.

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