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# SOLUTION OF A CAUCHY TYPE PROBLEM FOR AN INTEGRAL EQUATION OF VOLTERRA TYPE WITH SINGULAR KERNELS, WHEN THE ROOTS OF THE CHARACTERISTIC EQUATIONS ARE COMPLEX CONJUGATE

**Abstract:** In this paper, we study a two-dimensional integral equation of Volterra type with a singularity and a logarithmic singularity in one variable and a strong singularity in another variable. Solving an integral equation with special kernels in the case when the coefficients of the equation are related to each other reduces to solving one-dimensional integral equations of Volterra type with special kernels. Using the connection between the considered integral equations and ordinary differential equations with singular coefficients, depending on the sign of the coefficients of the equation and the roots of the characteristic equations, explicit solutions of the studied two-dimensional integral equation are obtained.

Note that explicit solutions of a two-dimensional integral equation of Volterra type with a singularity and a logarithmic singularity in one variable and a strong singularity in another variable can contain from one to four arbitrary functions. Cases have also been established when the solution to a two-dimensional integral equation of Volterra type with a singularity and a logarithmic singularity in one variable and a strong singularity in another variable is unique.

If the characteristic equations have complex conjugate roots, then the given integral equation with singular kernels has a unique solution or the explicit solutions contain two or four arbitrary functions. In the latter cases, the correct formulation was clarified and explicit solutions of Cauchy type problems were obtained.

**Keywords:** two-dimensional integral equation, singular line, logarithmic singularity, strongly singular kernel, differential equation, singular coefficients, complex conjugate roots.

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## INTRODUCTION

One of the well-known ways to study various problems of mathematical physics is the theory of singular integral equations. One-dimensional and multidimensional, linear and nonlinear integral equations of Volterra type with a singular kernel are often encountered in solving various applied problems in mathematics, physics, elasticity theory, and the theory of conformal mappings.

Note that the work of A.P. Soldatov [1] is devoted to the study of the characteristic integral equation with the Cauchy kernel, obtaining the solvability condition and an explicit formula for representing the solution. Singular integral equations with a singularity of logarithmic or power type, also simultaneously with weak and strong singularities in various combinations, were studied in [2]. Works [3-6] are devoted to the study of weakly singular and singular integral equations of various types, the construction and justification of computational schemes for the studied equations.

The works of N. Radjabov [7-8] are devoted to obtaining explicit varieties of solutions and studying boundary value problems of the Cauchy type of integral equations of Volterra type with a fixed boundary and internal singular, super-singular kernel, as well as integral equations with a logarithmic and singular singularity of the kernel.

Note that in [9-12] explicit manifolds of solutions of a two-dimensional integral equation of Volterra type (1) with a logarithmic and singular singularity in one of the variables and a strong boundary fixed singularity in the other variable were obtained, in the case when the coefficients of the equation are related to each other, also the roots of characteristic equations are real-different, real-equal, complex conjugate. In [13], the solution to the Volterra type integral equation with singular kernels (1) was obtained in the form of generalized functional series.

Note that in [14]-[15] problems of Cauchy type were studied, when the roots of the characteristic equations of a two-dimensional integral equation with singular kernels (1) were real and different, real and equal.

The proposed work is devoted to the formulation and solution of Cauchy type problems for determining arbitrary functions in the resulting solution, when the roots of the characteristic equations of the integral equation (1) are complex conjugate.

Let $D$ be a rectangle $D = \{(x,y) : a < x < a_1, b < y < b_1\}$ with boundaries $\Gamma_1 = \{y = b,\ a < x < a_1\}$, $\Gamma_2 = \{x = a,\ b < y < b_1\}$.

In the domain $D$ we study an integral equation of Volterra type with special kernels of the form:

$$u(x,y) + \int_a^x \left[p + q\ln\left(\frac{x-a}{t-a}\right)\right]\frac{u(t,y)}{t-a}dt + \int_b^y \left[\lambda + \mu\big(\omega_b^\beta(s) - \omega_b^\beta(y)\big)\right]\frac{u(x,s)}{(s-b)^\beta}ds +$$
$$+\int_a^x \left[p_1 + q_1\ln\left(\frac{x-a}{t-a}\right)\right]\frac{dt}{t-a}\int_b^y \left[\lambda_1 + \mu_1\big(\omega_b^\beta(s) - \omega_b^\beta(y)\big)\right]\frac{u(t,s)}{(s-b)^\beta}ds = f(x,y), \qquad (1)$$

where $p, q, \lambda, \mu, p_1, q_1, \lambda_1, \mu_1$ — given constants, $f(x,y)$ — given function, $u(x,y)$ — sought function, $\omega_b^\beta(s) = \left[(\beta-1)(y-b)^{\beta-1}\right]^{-1}, \beta > 1$.

We will look for a solution to the integral equation (1) in the class of functions $u(x,y) \in C(\overline{D})$ vanishing on $\Gamma_1$ and $\Gamma_2$ with asymptotic behavior:

$$u(x,y) = o[(x-a)^\varepsilon],\ \varepsilon > 0 \text{ at } x \to a,$$
$$u(x,y) = o[(y-b)^\nu],\ \nu > 2(\beta-1) \text{ at } y \to b.$$

**Problem $\mathbf{K_1}$.** Need to find a solution integral equation with special kernels (1) from class $C(\overline{D})$, vanishing on $\Gamma_1$ and $\Gamma_2$, subject to the conditions $p < 0, \lambda < 0, \Delta_1 = p^2 - 4q < 0, \Delta_2 = \lambda^2 - 4\mu < 0, p = p_1, q = q_1, \lambda = \lambda_1, \mu = \mu_1$, also conditions:

$$\begin{cases} \left\{(x-a)^{-\frac{|p|}{2}}\left[\left(\frac{|p|}{2}\sin\big(\Omega(x)\big) + \frac{\sqrt{4q-p^2}}{2}\cos\big(\Omega(x)\big)\right)u(x,y) - \sin\big(\Omega(x)\big)D_x\big(u(x,y)\big)\right]\right\}_{x=a} = A_1(y), \\ \left\{(x-a)^{-\frac{|p|}{2}}\left[\left(-\frac{|p|}{2}\cos\big(\Omega(x)\big) + \frac{\sqrt{4q-p^2}}{2}\sin\big(\Omega(x)\big)\right)u(x,y) + \cos\big(\Omega(x)\big)D_x\big(u(x,y)\big)\right]\right\}_{x=a} = A_2(y), \\ \left\{e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\left[\left(-\frac{\lambda}{2}\sin\big(\Theta(y)\big) - \frac{\sqrt{4\mu-\lambda^2}}{2}\cos\big(\Theta(y)\big)\right)u(x,y) - \sin\big(\Theta(y)\big)D_y\big(u(x,y)\big)\right]\right\}_{y=b} = B_1(x), \\ \left\{e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\left[\left(\frac{\lambda}{2}\cos\big(\Theta(y)\big) - \frac{\sqrt{4\mu-\lambda^2}}{2}\sin\big(\Theta(y)\big)\right)u(x,y) + \cos\big(\Theta(y)\big)D_y\big(u(x,y)\big)\right]\right\}_{y=b} = B_2(x), \end{cases}$$

where $B_1(x), B_2(x), A_1(y), A_2(y)$ are given continuous functions, $\Theta(y) = \frac{\sqrt{4\mu-\lambda^2}}{2}\omega_b^\beta(y), \Omega(x) = \frac{\sqrt{4q-p^2}}{2}\ln(x-a)$.

**Solution to problem $\mathbf{K_1}$.**

Solution of integral equation (1) at $p < 0, \lambda < 0, \Delta_1 = p^2 - 4q < 0, \Delta_2 = \lambda^2 - 4\mu < 0, p = p_1, q = q_1, \lambda = \lambda_1, \mu = \mu_1$, according to [6], has the form:

$$u(x,y) = (x-a)^{\frac{|p|}{2}}\left\{\cos\Big(\Omega(x)\Big)\theta_1(y) + \sin\Big(\Omega(x)\Big)\theta_2(y)\right\} +$$
$$+e^{\frac{\lambda}{2}\omega_b^\beta(y)}\left\{\cos\Big(\Theta(y)\Big)\Phi_1(x) + \sin\Big(\Theta(y)\Big)\Phi_2(x)\right\} + E^{(x,y)}_{p,q,\lambda,\mu}\left[f(x,y)\right], \qquad (2)$$

where

$$\Phi_i(x) = \varphi_i(x) + \frac{1}{\sqrt{4q-p^2}}\times$$

$$\times\int_a^x \left(\frac{x-a}{t-a}\right)^{\frac{|p|}{2}} \left[(p^2-4q)\sin\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right) - p\sqrt{4q-p^2}\cos\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right)\right]\frac{\varphi_i(x)dt}{t-a},$$

$\varphi_i(x), \theta_i(y), i=1,2$ are arbitrary continuous functions of the points $\Gamma_1$ and $\Gamma_2$,

$$E^{(x,y)}_{p,q,\lambda,\mu}[f(x,y)] = f(x,y) + \frac{1}{\sqrt{4q-p^2}}\times$$
$$\times\int_a^x \left(\frac{x-a}{t-a}\right)^{\frac{|p|}{2}} \left[(p^2-4q)\sin\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right) - p\sqrt{4q-p^2}\cos\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right)\right]\frac{f(t,y)}{t-a}dt+$$
$$+\frac{2}{\sqrt{4\mu-\lambda^2}}\int_b^y e^{\frac{\lambda}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))}\times$$
$$\times\left[\left(\frac{\lambda^2-2\mu}{2}\right)\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))\right) + \frac{\lambda\sqrt{4\mu-\lambda^2}}{2}\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))\right)\right]\frac{f(x,s)ds}{(s-b)^\beta}+$$
$$+\frac{1}{\sqrt{4q-p^2}\sqrt{4\mu-\lambda^2}}\int_b^y e^{\frac{\lambda}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))}\times$$
$$\times\left[\left(\frac{\lambda^2-2\mu}{2}\right)\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))\right) + \frac{\lambda\sqrt{4\mu-\lambda^2}}{2}\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\omega_b^\beta(y)-\omega_b^\beta(s))\right)\right]\frac{ds}{(s-b)^\beta}\times$$
$$\times\int_a^x \left(\frac{x-a}{t-a}\right)^{\frac{|p|}{2}} \left[(p^2-4q)\sin\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right) - p\sqrt{4q-p^2}\cos\left(\frac{\sqrt{4q-p^2}}{2}\ln\left(\frac{x-a}{t-a}\right)\right)\right]\frac{f(t,s)}{t-a}dt.$$

Let's represent solution (2) in the form:

$$u(x,y) = (x-a)^{\frac{|p|}{2}}\left\{\cos\Big(\Omega(x)\Big)\theta_1(y) + \sin\Big(\Omega(x)\Big)\theta_2(y)\right\} + T_1[\varphi_1(x),\varphi_2(x),f(x,y)], \qquad (3)$$

where

$$T_1[\varphi_1(x),\varphi_2(x),f(x,y)] = e^{\frac{\lambda}{2}\omega_b^\beta(y)}\left\{\cos\Big(\Theta(y)\Big)\Phi_1(x) + \sin\Big(\Theta(y)\Big)\Phi_2(x)\right\} + E^{(x,y)}_{p,q,\lambda,\mu}[f(x,y)].$$

Differentiating equality (3), then multiplying by $x-a$, we obtain:

$$D_x u(x,y) = (x-a)\frac{du(x,y)}{dx} = (x-a)^{\frac{|p|}{2}}\times$$
$$\times\left\{\left[\frac{|p|}{2}\cos\Big(\Omega(x)\Big) - \frac{\sqrt{4q-p^2}}{2}\sin\Big(\Omega(x)\Big)\right]\theta_1(y) + \left[\frac{|p|}{2}\sin\Big(\Omega(x)\Big) + \frac{\sqrt{4q-p^2}}{2}\cos\Big(\Omega(x)\Big)\right]\theta_2(y)\right\}+$$
$$+D_x T_1[\varphi_1(x),\varphi_2(x),f(x,y)].$$

From the obtained equalities for defining arbitrary functions $\theta_1(y)$ and $\theta_2(y)$ we obtain a linear algebraic system of equations of the form:

$$\begin{cases}\cos(\Omega(x))\,\theta_1(y) + \sin(\Omega(x))\,\theta_2(y) = (x-a)^{-\frac{|p|}{2}}\big[u(x,y) - T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big],\\ \left[\frac{|p|}{2}\cos(\Omega(x)) - \frac{\sqrt{4q-p^2}}{2}\sin(\Omega(x))\right]\theta_1(y) + \left[\frac{|p|}{2}\sin(\Omega(x)) + \frac{\sqrt{4q-p^2}}{2}\cos(\Omega(x))\right]\theta_2(y) =\\ = (x-a)^{-\frac{|p|}{2}}\big[D_x u(x,y) - D_x T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big].\end{cases}$$

To determine the unknowns $\theta_1(y)$ and $\theta_2(y)$ we use Cramer's method:

$$\Delta = \begin{vmatrix} \cos\big(\Omega(x)\big) & \sin\big(\Omega(x)\big) \\ \frac{|p|}{2}\cos\big(\Omega(x)\big) - \frac{\sqrt{4q-p^2}}{2}\sin\big(\Omega(x)\big) & \frac{|p|}{2}\sin\big(\Omega(x)\big) + \frac{\sqrt{4q-p^2}}{2}\cos\big(\Omega(x)\big) \end{vmatrix} = \frac{\sqrt{4q-p^2}}{2},$$

$$\Delta_{\theta_1(y)} = \begin{vmatrix} (x-a)^{-\frac{|p|}{2}}\big[u(x,y) - T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big] & \sin\big(\Omega(x)\big) \\ (x-a)^{-\frac{|p|}{2}}\big[D_x u(x,y) - D_x T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big] & \frac{|p|}{2}\sin\big(\Omega(x)\big) + \frac{\sqrt{4q-p^2}}{2}\cos\big(\Omega(x)\big) \end{vmatrix} =$$

$$= \frac{\sqrt{4q-p^2}}{2}\theta_1(y),$$

$$\Delta_{\theta_2(y)} = \begin{vmatrix} \cos\big(\Omega(x)\big) & (x-a)^{-\frac{|p|}{2}}\big[u(x,y) - T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big] \\ \frac{|p|}{2}\cos\big(\Omega(x)\big) - \frac{\sqrt{4q-p^2}}{2}\sin\big(\Omega(x)\big) & (x-a)^{-\frac{|p|}{2}}\big[D_x u(x,y) - D_x T_1[\varphi_1(x),\varphi_2(x),f(x,y)]\big] \end{vmatrix} =$$

$$= \frac{\sqrt{4q-p^2}}{2}\theta_2(y).$$

From here we find:

$$\begin{cases} \theta_1(y) = \frac{\Delta_{\theta_1(y)}}{\Delta} = \theta_1(y) \equiv A_1(y), \\ \theta_2(y) = \frac{\Delta_{\theta_2(y)}}{\Delta} = \theta_2(y) \equiv A_2(y). \end{cases}$$

In a similar way, we represent solution (2) in the form:

$$u(x,y) = e^{\frac{\lambda}{2}\omega_b^\beta(y)}\Big\{\cos\Big(\Theta(y)\Big)\Phi_1(x) + \sin\Big(\Theta(y)\Big)\Phi_2(x)\Big\} + T_2[\theta_1(y),\theta_2(y),f(x,y)], \tag{4}$$

where

$$T_2[\theta_1(y),\theta_2(y),f(x,y)] = (x-a)^{\frac{|p|}{2}}\Big\{\cos\Big(\Omega(x)\Big)\theta_1(y) + \sin\Big(\Omega(x)\Big)\theta_2(y)\Big\} + E^{(x,y)}_{p,q,\lambda,\mu}\left[f(x,y)\right].$$

Calculating the derivative by (4), then multiplying by $(y-b)^\beta$, we obtain:

$$D_y u(x,y) = (y-b)^\beta \frac{du(x,y)}{dy} = e^{\frac{\lambda}{2}\omega_b^\beta(y)} \times$$

$$\times\Bigg\{\left[\frac{\sqrt{4\mu-\lambda^2}}{2}\sin\Big(\Theta(y)\Big) - \frac{\lambda}{2}\cos\Big(\Theta(y)\Big)\right]\Phi_1(x) - \left[\frac{\sqrt{4\mu-\lambda^2}}{2}\cos\Big(\Theta(y)\Big) + \frac{\lambda}{2}\sin\Big(\Theta(y)\Big)\right]\Phi_2(x)\Bigg\} +$$

$$+ D_y T_2[\theta_1(y),\theta_2(y),f(x,y)].$$

From the obtained equalities for determining the functions $\Phi_1(x)$ and $\Phi_2(x)$ we obtain a linear algebraic system of equations of the form:

$$\begin{cases} \cos\Big(\Theta(y)\Big)\Phi_1(x) + \sin\Big(\Theta(y)\Big)\Phi_2(x) = e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[u(x,y) - T_2[\theta_1(y),\theta_2(y),f(x,y)]\big], \\ \left[\frac{\sqrt{4\mu-\lambda^2}}{2}\sin\Big(\Theta(y)\Big) - \frac{\lambda}{2}\cos\Big(\Theta(y)\Big)\right]\Phi_1(x) - \left[\frac{\sqrt{4\mu-\lambda^2}}{2}\cos\Big(\Theta(y)\Big) + \frac{\lambda}{2}\sin\Big(\Theta(y)\Big)\right]\Phi_2(x) = \\ = e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[D_y u(x,y) - D_y T_2[\theta_1(y),\theta_2(y),f(x,y)]\big]. \end{cases}$$

To solve the system of equations we use the Cramer method:

$$\Delta = \begin{vmatrix} \cos\big(\Theta(y)\big) & \sin\big(\Theta(y)\big) \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\sin\Big(\Theta(y)\Big) - \frac{\lambda}{2}\cos\Big(\Theta(y)\Big) & -\frac{\sqrt{4\mu-\lambda^2}}{2}\cos\Big(\Theta(y)\Big) - \frac{\lambda}{2}\sin\Big(\Theta(y)\Big) \end{vmatrix} = -\frac{\sqrt{4\mu-\lambda^2}}{2},$$

$$\Delta_{\Phi_1(x)} = \begin{vmatrix} e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[u(x,y) - T_2[\theta_1(y),\theta_2(y),f(x,y)]\big] & \sin\big(\Theta(y)\big) \\ e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[D_y u(x,y) - D_y T_2[\theta_1(y),\theta_2(y),f(x,y)]\big] & -\frac{\sqrt{4\mu-\lambda^2}}{2}\cos\Big(\Theta(y)\Big) - \frac{\lambda}{2}\sin\Big(\Theta(y)\Big) \end{vmatrix} =$$

$$= -\frac{\sqrt{4\mu-\lambda^2}}{2}\Phi_1(x),$$

$$\Delta_{\Phi_2(x)} = \begin{vmatrix} \cos\big(\Theta(y)\big) & e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[u(x,y) - T_2[\theta_1(y),\theta_2(y),f(x,y)]\big] \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\sin\Big(\Theta(y)\Big) - \frac{\lambda}{2}\cos\Big(\Theta(y)\Big) & e^{-\frac{\lambda}{2}\omega_b^\beta(y)}\big[D_y u(x,y) - D_y T_2[\theta_1(y),\theta_2(y),f(x,y)]\big] \end{vmatrix} =$$

$$= -\frac{\sqrt{4\mu-\lambda^2}}{2}\Phi_2(x).$$

From here we find:

$$\begin{cases}\Phi_1(x)=\frac{\Delta_{\Phi_1(x)}}{\Delta}=\Phi_1(x)\equiv B_1(x),\\ \Phi_2(x)=\frac{\Delta_{\Phi_2(x)}}{\Delta}=\Phi_2(x)\equiv B_2(x).\end{cases}$$

MAIN RESULTS

From the above considerations the following statement follows about the solvability of the problem $K_1$ :

**Theorem 1.** *Let the coefficients in the integral equation (1) satisfy the conditions of the problem $K_1$, the right side satisfy the conditions $f(x,y)\in C(\overline{D})$ , $f(a,b)=0$ with asymptotic behavior on $\Gamma_1$ and $\Gamma_2$ :*

$$f(x,y)=o[(x-a)^{\delta}],\ \delta>\frac{|p|}{2}\text{ at }x\to a,$$
$$f(x,y)=o[e^{\frac{\lambda}{2}\omega_b^{\beta}(y)}(y-b)^{\nu}],\ \nu>2(\beta-1)\text{ at }y\to b.$$

*Then problem $K_1$ has a unique solution, which is expressed by the equality:*

$$u(x,y)=(x-a)^{\frac{|p|}{2}}\left\{\cos\Big(\Omega(x)\Big)A_1(y)+\sin\Big(\Omega(x)\Big)A_2(y)\right\}+$$
$$+e^{\frac{\lambda}{2}\omega_b^{\beta}(y)}\left\{\cos\Big(\Theta(y)\Big)B_1(x)+\sin\Big(\Theta(y)\Big)B_2(x)\right\}+E_{p,q,\lambda,\mu}^{(x,y)}\left[f(x,y)\right].$$

**Problem $\mathbf{K_2}$ .** Need to find a solution integral equation with special kernels (1) from class $C(\overline{D})$ , vanishing on $\Gamma_1$ and $\Gamma_2$ , subject to the conditions $p<0$ , $\lambda>0$ , $\Delta_1=p^2-4q<0$ , $\Delta_2=\lambda^2-4\mu<0$ , $p=p_1$, $q=q_1,\lambda=\lambda_1$, $\mu=\mu_1$ , also conditions:

$$\begin{cases}l\left\{(x-a)^{-\frac{|p|}{2}}\left[\left(\frac{|p|}{2}\sin\big(\Omega(x)\big)+\frac{\sqrt{4q-p^2}}{2}\cos\big(\Omega(x)\big)\right)u(x,y)-\sin\big(\Omega(x)\big)D_x\big(u(x,y)\big)\right]\right\}_{x=a}=A_1(y),\\ \left\{(x-a)^{-\frac{|p|}{2}}\left[\left(-\frac{|p|}{2}\cos\big(\Omega(x)\big)+\frac{\sqrt{4q-p^2}}{2}\sin\big(\Omega(x)\big)\right)u(x,y)+\cos\big(\Omega(x)\big)D_x\big(u(x,y)\big)\right]\right\}_{x=a}=A_2(y),\end{cases}$$

where $A_1(y)$ , $A_2(y)$ – given continuous functions.

To solve problem $K_2$ using the method for solving problem $K_1$ , we arrive at the following statement:

**Theorem 2.** *Let the coefficients in the integral equation (1) satisfy the conditions of the problem $K_2$ , the right side satisfy the conditions $f(x,y)\in C(\overline{D})$ , $f(a,b)=0$ with asymptotic behavior on $\Gamma_1$ and $\Gamma_2$ :*

$$f(x,y)=o[(x-a)^{\delta_1}],\ \delta_1>\frac{|p|}{2}\text{ at }x\to a,$$
$$f(x,y)=o[(y-b)^{\nu_1}],\ \nu_1>2(\beta-1)\text{ at }y\to b.$$

*Then problem $K_2$ has a unique solution, which is expressed by the equality:*

$$u(x,y)=(x-a)^{\frac{|p|}{2}}\left\{\cos\Big(\Omega(x)\Big)A_1(y)+\sin\Big(\Omega(x)\Big)A_2(y)\right\}+E_{p,q,\lambda,\mu}^{(x,y)}\left[f(x,y)\right].$$

**Problem $\mathbf{K_3}$ .** Need to find a solution integral equation with special kernels (1) from class $C(\overline{D})$ , vanishing on $\Gamma_1$ and $\Gamma_2$ , subject to the conditions $p>0$ , $\lambda<0$ , $\Delta_1=p^2-4q<0$ , $\Delta_2=\lambda^2-4\mu<0$ , $p=p_1$, $q=q_1$, $\lambda=\lambda_1$, $\mu=\mu_1$ , also conditions:

$$\begin{cases}\left\{e^{-\frac{\lambda}{2}\omega_b^{\beta}(y)}\left[\left(-\frac{\lambda}{2}\sin\big(\Theta(y)\big)-\frac{\sqrt{4\mu-\lambda^2}}{2}\cos\big(\Theta(y)\big)\right)u(x,y)-\sin\big(\Theta(y)\big)D_x\big(u(x,y)\big)\right]\right\}_{y=b}=B_1(x),\\ \left\{e^{-\frac{\lambda}{2}\omega_b^{\beta}(y)}\left[\left(\frac{\lambda}{2}\cos\big(\Theta(y)\big)-\frac{\sqrt{4\mu-\lambda^2}}{2}\sin\big(\Theta(y)\big)\right)u(x,y)+\cos\big(\Theta(y)\big)D_x\big(u(x,y)\big)\right]\right\}_{y=b}=B_2(x),\end{cases}$$

where $B_1(x)$ , $B_2(x)$ are given continuous functions.

The statement is true:

**Theorem 3.** *Let the coefficients in the integral equation (1) satisfy the conditions of the problem $K_3$ , the right side satisfy the conditions $f(x,y)\in C(\overline{D})$ , $f(a,b)=0$ with asymptotic behavior on $\Gamma_1$ and $\Gamma_2$ :*

$$f(x,y)=o[(x-a)^{\varepsilon}],\ \varepsilon>0\text{ at }x\to a,$$
$$f(x,y)=o[e^{\frac{\lambda}{2}\omega_b^{\beta}(y)}(y-b)^{\nu_2}],\ \nu_2>2(\beta-1)\text{ at }y\to b.$$

*Then problem* $K_3$ *has a unique solution, which is expressed by the equality:*

$$u(x,y) = e^{\frac{\lambda}{2}\omega_b^\beta(y)} \left\{ \cos\Big(\Theta(y)\Big) B_1(x) + \sin\Big(\Theta(y)\Big) B_2(x) \right\} + E^{(x,y)}_{p,q,\lambda,\mu}\left[f(x,y)\right].$$

CONCLUSION

In [9-12], explicit varieties of solutions to a two-dimensional integral equation of Volterra type with special and strongly special fixed boundary lines were obtained, which, depending on the signs of the coefficients of the equation and the roots of the characteristic equations, have from one to four arbitrary functions depending on one variable. The case is highlighted when the solution to a two-dimensional integral equation of Volterra type with singular and strongly singular kernels is unique, coinciding with the classical theory of integral equations with a regular kernel or a kernel with a weak singularity. In this work, in the case where the roots of the characteristic equations are complex conjugate, Cauchy type problems are posed and solved to determine arbitrary functions in the resulting solutions.

Authors' contribution

The formulation of the problem and the method of solution were proposed by L.N. Rajabova, all calculations performed by F.M. Akhmadov.

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**Сипаттамалық теңдеулердің шешімдері комплексті-түйіндес болған жағдайда ерекше ядролы Вольтерра типті интегралдық теңдеулер үшін Коши типті есептің шешімі**

**Аннотация:** Мақалада бірінші айнымалысы бойынша ерекшеліктері және логарифмдік ерекшеліктері, екінші айнымалы бойынша қатты ерекшелігі бар Вольтерра типті екі өлшемді интегралдық теңдеу қарастырылады. Теңдеу шешімдері өзара байланысқан болған жағдайда ерекше ядролы интегралдық теңдеулер ерекше ядролы Вольтерра типті бірөлшемді интегралдық теңдеулерді шешуге әкелінеді. Теңдеудің коэффициенттерінің таңбалары мен сипаттамалық теңдеу шешімдеріне сәйкес қарастырылып отырған интегралдық теңдеулер мен сингулярлы коэффициентті жай дифференциалдық теңдеулер арасындағы байланысты қолдана отырып, екі өлшемді интегралдық теңдеудің айқын шешімдері табылды.

Ерекшеліктері және бірінші айнымалы бойынша логарифмдік, бірінші айнымалысы бойынша ерекшеліктері және логарифмдік ерекшеліктері, екінші айнымалы бойынша қатты ерекшелігі бар Вольтерра типті екі өлшемді интегралдық теңдеудің айқын шешімдері бірден төртке дейін кездейсоқ функцияларды қамтуы мүмкін екендігін атап өтейік. Ерекшеліктері және бірінші айнымалы бойынша логарифмдік, бірінші айнымалысы бойынша ерекшеліктері және логарифмдік ерекшеліктері, екінші айнымалы бойынша қатты ерекшелігі бар Вольтерра типті екі өлшемді интегралдық теңдеудің шешімдері жалғыз болатын жағдайлар да кездесті.

Егер сипаттамалық теңдеулердің комплекті-түйіндес шешімдері бар болса, онда ерекше ядролы интегралдық теңдеудің жалғыз шешімі болады немесе айқын шешімдері екі немесе төрт кездейсоқ функцияны қамтиды. Соңғы жағдайларда есеп қойылымының дұрыстығы анықталып, Коши типті есептің айқын шешімдері алынды

**Түйін сөздер:** екі өлшемді интегралдық теңдеу, ерекше сызық, логарифмдік ерешелік, күшті-ерекше ядро, дифференциалдық теңдеу, сингулярлы коэффициенттер, комплексті-түйіндес шешімдер.

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**Решение задачи типа Коши для интегрального уравнения типа Вольтерра с особыми ядрами, когда корни характеристических уравнений комплексно-сопряжённые**

**Абстракт:** В данной работе изучается двумерное интегральное уравнение типа Вольтерра с особенностью и логарифмической особенностью по одной переменной и сильной особенностью по другой переменной. Решение интегрального уравнения с особыми ядрами в случае, когда коэффициенты уравнения связаны между собой, сводится к решению одномерных интегральных уравнений типа Вольтерра с особыми ядрами. Используя связь рассматриваемых интегральных уравнений с обыкновенными дифференциальными уравнениями с сингулярными коэффициентами, в зависимости от знака коэффициентов уравнения и корней характеристических уравнений, получены явные решения изучаемого двумерного интегрального уравнения.

Отметим, что явные решения двумерного интегрального уравнения типа Вольтерра с особенностью и логарифмической особенностью по одной переменной и сильной особенностью по другой переменной может содержать от одного до четырех произвольных функций. Также установлены случаи, когда решение двумерного интегрального уравнения типа Вольтерра с особенностью и логарифмической особенностью по одной переменной и сильной особенностью по другой переменной единственно.

Если характеристические уравнения имеют комплексно-сопряжённые корни, тогда данное интегральное уравнение с особыми ядрами имеет единственное решение или явные решения содержат две или четыре произвольные функции. В последних случаях выяснена корректная постановка и получены явные решения задач типа Коши.

**Ключевые слова:** двумерное интегральное уравнение, особая линия, логарифмическая особенность, сильно-особое ядро, дифференциальное уравнение, сингулярные коэффициенты, комплексно-сопряжённые корни.

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