ISSN 2616-7182

Л.Н. Гумилев атындағы Еуразия ұлттық университетінің

ХАБАРШЫСЫ

BULLETIN
of the L.N. Gumilyov Eurasian
National University

ВЕСТНИК
Евразийского национального
университета имени Л.Н. Гумилева

МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА сериясы

MATHEMATICS. COMPUTER SCIENCE. MECHANICS Series

Серия **МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА**

№1(122)/2018

1995 жылдан бастап шығады

Founded in 1995

Издается с 1995 года

Жылына 4 рет шығады
Published 4 times a year
Выходит 4 раза в год

Астана, 2018
Astana, 2018

Редакцияның мекенжайы: 010008, Қазақстан, Астана қ., Сәтпаев к-сі, 2, 408 бөлме.
Тел: (7172) 709-500 (ішкі 31-428). E-mail: *vest_math@enu.kz*

Жауапты хатшы, компьютерде беттеген
А. Нұрболат

Л.Н. Гумилев атындағы Еуразия ұлттық университетінің хабаршысы.
МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА сериясы
Меншіктенуші: ҚР БжҒМ "Л.Н. Гумилев атындағы Еуразия ұлттық университеті" ШЖҚ РМК
Мерзімділігі: жылына 4 рет.
Қазақстан Республикасының Ақпарат және коммуникациялар министрлігімен тіркелген.
27.03.2018ж. № 17000-ж тіркеу куәлігі.
Тиражы: 30 дана
Типографияның мекенжайы: 010008, Қазақстан, Астана қ., Қажымұқан к-сі ,12/1,
тел: (7172)709-500 (ішкі 31-428).

Editorial address: 2, Satpayev str., of. 408, Astana, Kazakhstan, 010008
Tel.: (7172) 709-500 (ext. 31-428)
E-mail: *vest_math@enu.kz*

Responsible secretary, computer layout:
A. Nurbolat

Bulletin of the L.N. Gumilyov Eurasian National University.
MATHEMATICS. COMPUTER SCIENCE. MECHANICS Series
Owner: Republican State Enterprise in the capacity of economic conduct "L.N. Gumilyov Eurasian National University" Ministry of Education and Science of the Republic of Kazakhstan
Periodicity: 4 times a year
Registered by the Ministry of Information and Communication of the Republic of Kazakhstan.
Registration certificate №17000-ж from 27.03.2018.
Circulation: 30 copies
Address of printing house: 12/1 Kazhimukan str., Astana, Kazakhstan 010008;
tel: (7172) 709-500 (ext.31-428).

Адрес редакции: 010008, Казахстан, г. Астана, ул. Сатпаева, 2, каб. 408
Тел: (7172) 709-500 (вн. 31-428). E-mail: *vest_math@enu.kz*

Ответственный секретарь, компьютерная верстка
А. Нурболат

Вестник Евразийского национального университета имени Л.Н. Гумилева.
Серия МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА
Собственник: РГП на ПХВ "Евразийский национальный университет имени Л.Н. Гумилева" МОН РК
Периодичность: 4 раза в год.
Зарегистрирован Министерством информации и коммуникаций Республики Казахстан.
Регистрационное свидетельство №17000-ж от 27.03.2018г.
Тираж: 30 экземпляров. Адрес типографии: 010008, Казахстан, г. Астана, ул. Кажымукана, 12/1,
тел.: (7172)709-500 (вн.31-428).

Л.Н. ГУМИЛЕВ АТЫНДАҒЫ ЕУРАЗИЯ ҰЛТТЫҚ УНИВЕРСИТЕТІНІҢ ХАБАРШЫСЫ. МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА СЕРИЯСЫ

№1(122)/2018

МАЗМҰНЫ

**BULLETIN OF L.N. GUMILYOV EURASIAN NATIONAL UNIVERSITY.
MATHEMATICS. COMPUTER SCIENCE. MECHANICS SERIES**

№1(122)/2018

CONTENTS

ВЕСТНИК ЕВРАЗИЙСКОГО НАЦИОНАЛЬНОГО УНИВЕРСИТЕТА ИМЕНИ Л.Н.ГУМИЛЕВА. СЕРИЯ МАТЕМАТИКА. ИНФОРМАТИКА. МЕХАНИКА
№1(122)/2018

СОДЕРЖАНИЕ

Л.Н. Гумилев атындағы Еуразия ұлттық университетінің хабаршысы. Математика. Информатика. Механика сериясы, 2018, 1(122), 76-89 беттер
http://bulmathmc.enu.kz, E-mail: vest_math@enu.kz

МРНТИ: **27.25.19**

Dinh Dũng

Information Technology Institute, Vietnam National University, 144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
(E-mail: dinhzung@gmail.com)

Galerkin approximation for parametric and stochastic elliptic PDEs

Abstract: We study the Galerkin approximation for the parametric elliptic problem

$$-\operatorname{div}\big(a(y)(x)\nabla u(y)(x)\big) \ = \ f(x) \quad x \in D, \ y \in \mathbb{I}^\infty, \quad u|_{\partial D} \ = \ 0,$$

where $D \subset \mathbb{R}^m$ is a bounded Lipschitz domain, $\mathbb{I}^\infty := [-1,1]^\infty$, $f \in L_2(D)$, and the diffusions a satisfy the uniform ellipticity assumption and are affinely dependent with respect to y. Assume that we have an approximation property that there is a sequence of finite element approximations with a certain error convergence rate in energy norm of the space $V := H^1_0(D)$ for the nonparametric problem $-\operatorname{div}\big(a(y_0)(x)\nabla u(y_0)(x)\big) = f(x)$ at almost every point $y_0 \in \mathbb{I}^\infty$ with regard to the uniform probability measure μ on $\mathbb{I}^\infty$. Based on this assumption we construct a sequence of finite element approximations with the same error convergence rate for the parametric elliptic problem in the norm of the Bochner spaces $L_2(\mathbb{I}^\infty, V, \mu)$. This shows that the curse of dimensionality for the parametric elliptic problem is rid by linear methods.

Keywords: Parametric and stochastic elliptic PDEs, collective Galerkin approximation, affine dependence of the diffusion coefficients, the curse of dimensionality.

1. Introduction

Let $D \subset \mathbb{R}^m$ be a bounded domain with a Lipschitz boundary ∂D and $\mathbb{I}^d := [-1,1]^d$. Consider the parametric elliptic problem

$$-\operatorname{div}(a(y)\nabla u(y)) \ = \ f \quad \text{in} \quad D, \quad u|_{\partial D} \ = \ 0, \quad y \in \mathbb{I}^d, \tag{1}$$

where the gradient operator ∇ is taken with respect to x, the diffusions $a(y)(x) := a(x,y)$ are functions of $x = (x_1, ..., x_m) \in D$ and of parameters $y = (y_1, ..., y_d) \in \mathbb{I}^\infty$ on $D \times \mathbb{I}^d$, and the function $f(x)$ is functions of $x = (x_1, ..., x_m) \in D$. In the equation (1), the parameter y may be either deterministic or random variable. The main challenge in numerical computation is to approximate the entire solution map $y \mapsto u(y)$ up to a prescribed accuracy with acceptable cost. This problem becomes actually difficult when d may be very large. Here we suffer the so-called curse of dimensionality coined by Bellman: the computational cost grows exponentially in the dimension d of the parametric space. Moreover, in some models the number of parameters may be even countably infinite.

Based on finite element approximations with respect to the spatial variable x and polynomial approximations with respect to the parametric variable y, there have been proposed several numerical methods for solving (1). Many works have been devoted to the development of the parametric Garlerkin technique for the numerical solving of (1). As shown in [13], these methods are promising since they can use the possible regularity of the solution $u(y)$ with respect to the parameters y to achieve faster convergence than sampling methods like Monte Carlo. A parametric Garlerkin method is a projection technique over a set of orthogonal polynomials with respect to an appropriate probability measure [1, 2, 5, 8, 10, 11, 13, 16, 17, 19, 20, 21]. We refer the reader to [9, 18, 22] for surveys and bibliography on different aspects in study of approximation and numerical methods for the problem (1).

In [7]–[11], [19] with an assumption on the ℓ_p-summability $\big(\|\psi_j\|_{W^1_\infty(D)}\big)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ for some $0 < p < 1$ on the affine expansion (2), the authors proposed nonlinear n-term approximation methods in energy norm by establishing *a priori* the set of the n most useful infinite dimensional polynomials in Taylor expansion, Legendre polynomials expansion and Lagrange interpolation. The obtained n-term approximands then are approximated by finite element methods. The results of [10, 11] have been improved [3, 4]. Best n-term approximations of infinite-dimensional polynomial expansions providing a benchmark for convergence rates, are not constructive. In the recent papers [13, 14], we have considered a particular case of the equation (1) where $D = [0,1]^m$, with an *a priori* assumption that the solution possesses higher order mixed smoothnesses of Sobolev-Korobov type or of Sobolev-analytic type simultaneously on spatial variable x and parametric variable y. Applying results on hyperbolic cross approximation in infinite dimension, we constructed linear collective Galerkin methods on both variables x and y for approximation of the solution which give the same approximation rate in energy norm as for the approximation by Galerkin methods for solving the corresponding nonparametric elliptic problem the domain $[0,1]^m$.

Throughout the present paper we preliminarily assume that $d = \infty$, $f \in L_2(D)$ and the diffusions a satisfy the uniform ellipticity assumption

$$0 \ < \ r \ < \ a(y)(x) = a(x,y) \ \le \ R \ < \ \infty, \quad x \in D, \ y \in \mathbb{I}^\infty,$$

and are affinely dependent with respect to y, or more precisely,

$$a(y)(x) \ = \ \bar{a}(x) \ + \ \sum_{j=1}^{\infty} y_j \, \psi_j(x), \quad x \in D, \ y \in \mathbb{I}^\infty, \quad \bar{a}, \psi_j \in W^1_\infty(D), \tag{2}$$

where $W^1_\infty(D)$ is the space of functions v on D, equipped with the semi-norm and norm

$$|v|_{W^1_\infty(D)} := \ \max_{1 \le i \le m} \|\partial_{x_i} v\|_{L_\infty(D)}, \quad \|v\|_{W^1_\infty(D)} := \ \|v\|_{L_\infty(D)} + |v|_{W^1_\infty(D)}.$$

Let $V := H^1_0(D)$ and denote by W the subspace of V equipped with the norm

$$\|v\|_W := \ \|\Delta v\|_{L_2(D)}.$$

(This is a norm since if $v \in V$ and $\Delta v = 0$, then $v = 0$). Assume that we have the following approximation property on the spatial domain D: There are a nested sequence of subspaces $(V_n)_{n \in \mathbb{N}}$ in V, a sequence of linear bounded operators $(P_n)_{n \in \mathbb{N}}$ from V into V_n, and a number $0 < \alpha \le 1/m$ such that $\dim V_n \le n$ and

$$\|v - P_n(v)\|_V \ \le \ C_D \, n^{-\alpha} \, \|v\|_W, \quad \forall v \in W. \tag{3}$$

In the present paper, we propose collective Galerkin and Legendre approximations in the Bochner spaces $L_2(\mathbb{I}^\infty, V)$ and $L_\infty(\mathbb{I}^\infty, V)$ for solving (1), based on this approximation property and the Legendre polynomials expansion on the parametric domain $\mathbb{I}^\infty$. All the methods are linear and constructive. These approximations are of hyperbolic cross type (see [15] for a survey and bibliography on hypebolic cross approximation and applications). Moreover, they are collective with regard to spatial variable x and parametric variable y. This means that in constructing these methods, the m-variate spatial part and the infinite-variate parametric part are not separately but collectively treated. Assume that

$$\big(\|\psi_j\|_{W^1_\infty(D)}\big)_{j \in \mathbb{N}} \in \ell_{p(\alpha)}(\mathbb{N})$$

with $p(\alpha) := \frac{2}{1+2\alpha}$ for the collective Galerkin approximation, and with $p(\alpha) := \frac{1}{1+\alpha}$ for the collective Legendre approximation. Under this condition on the diffusions $a(y)$, we show that our methods give *the same convergence rate* $n^{-\alpha}$ of the error of the approximation of the solution of the nonparametric elliptic problem using the approximation property (3). All the conditions on the diffusions $a(y)$ in particular, the $\ell_{p(\alpha)}$-assumption do not affect the convergence rate of the approximation error, completely disappear from it and influence only the constant. This in particular, shows that the curse of dimensionality for the parametric elliptic problem is rid by linear methods. Notice also that the construction of linear collective approximations in the present paper is different from the construction of finite element approximations in [8], [10], [11], and from the construction of linear collective approximations in [13, 14]. See also the extended arXiv preprint [12] and [23] for new related results on collective Taylor and collocation approximations for the parametric elliptic problem (1) .

The outline of the present paper is the following. In Section 2 we process the construction and error estimation of collective Galerkin methods for solving (1). In Section 3 we process the construction and error estimation of collective Legendre methods for solving (1).

2. Galerkin approximation

2.1. **Nonparametric elliptic problem.** Let us preliminarily consider the nonparametric situation when we have only one equation:

$$-\operatorname{div}(a\nabla u) \;=\; f \quad \text{in} \quad D, \quad u|_{\partial D} \;=\; 0, \tag{4}$$

where f, a are functions on D, $f \in L_2(D)$ and a satisfies the ellipticity assumption

$$0 \;<\; r \;<\; a(x) \;\leq\; R \;<\; \infty, \quad x \in D.$$

By the well-known Lax-Milgram lemma, there exists a unique solution $u \in V$ in weak form which satisfies the variational equation

$$\int_D a(x)\nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x \;=\; \int_D f(x)\,v(x)\,\mathrm{d}x, \quad \forall v \in V.$$

From the embedding inequality $\|f\|_{V^*} \;\leq\; \|f\|_{L_2(D)}$ and the inequality $\|u\|_V \;\leq\; \frac{1}{r}\|f\|_{V^*}$ we have that

$$\|u\|_V \;\leq\; \frac{\|f\|_{L_2(D)}}{r}, \tag{5}$$

where $V^* = H^{-1}(D)$ denotes the dual of V.

If we assume that $a \in W^1_\infty(D)$, then the solution u of (4) is in W. Moreover, u satisfies the estimate

$$\|u\|_W \;\leq\; \frac{1}{r}\left(1 + \frac{|a|_{W^1_\infty(D)}}{r}\right)\|f\|_{L_2(D)}. \tag{6}$$

For construction of a collective Galerkin approximation we need an approximation property in the spatial domain in the following assumption.

Assumption (i): There are a nested sequence of subspaces $(V_n)_{n\in\mathbb{N}}$ in V, a sequence of linear uniformly bounded operators $(P_n)_{n\in\mathbb{N}}$ from V into V_n, and a number $0 < \alpha \leq 1/m$ such that $\dim V_n \leq n$ and

$$\|v - P_n(v)\|_V \;\leq\; C_D\, n^{-\alpha}\, \|v\|_W, \quad \forall v \in W, \tag{7}$$

where C_D is a constant which may depend on the domain D.

For example, classical error estimates [6] yield that the convergence rate in (7) with $\alpha = 1/m$ can be achieved by using Lagrange finite elements on quasi-uniform partitions. Throughout the remainder of the present paper, α is fixed and used only for denoting the convergence rate in Assumption (i).

Under Assumption (i) by Céa's lemma we have

$$\|u - u_n\|_V \;\leq\; \sqrt{\frac{R}{r}}\inf_{v\in V_n}\|u - v\|_V \;\leq\; \sqrt{\frac{R}{r}}\,\|u - P_n(u)\|_V \;\leq\; \sqrt{\frac{R}{r}}\,C_D\, n^{-\alpha},$$

where u_n is the Galerkin approximation which is the unique solution of the problem

$$\int_D a(x)\nabla u_n(x) \cdot \nabla v(x)\,\mathrm{d}x \;=\; \int_D f(x)\,v(x)\,\mathrm{d}x, \quad \forall v \in V_n.$$

For $k \in \mathbb{Z}_+$, we define

$$\delta_k(v) :=\; P_{2^k}(v) - P_{2^{k-1}}(v),\; k \in \mathbb{N}, \quad \delta_0(v) = P_0(v).$$

If Assumption (i) holds, then we can represent every $v \in W$ by the series

$$v \;=\; \sum_{k=0}^{\infty} \delta_k(v)$$

converging in V and satisfying the estimate

$$\|\delta_k(v)\|_V \;\leq\; (1 + 2^\alpha)C_D\, 2^{-\alpha k}\, \|v\|_W, \quad k \in \mathbb{Z}_+, \tag{8}$$

which follows from Assumption (i) and the inequality

$$\|\delta_k(v)\|_V \ \le \ \|v - P_{2^k}(v)\|_V + \|v - P_{2^{k-1}}(v)\|_V.$$

2.2. **Parametric elliptic problem.** Let us first reformulate the parametric equation (1) in the variational form. For every $y \in \mathbb{I}^\infty$, by the well-known Lax-Milgram lemma, there exists a unique solution $u(y) \in V$ in weak form which satisfies the variational equation

$$\int_D a(x,y)\nabla u(y)(x) \cdot \nabla v(x)\,\mathrm{d}x \ = \ \int_D f(x)\,v(x)\,\mathrm{d}x, \quad \forall v \in V. \tag{9}$$

Define a probability measure μ on $\mathbb{I}^\infty$ as the infinite tensor product measure of the univariate uniform probability measures on the one-dimensional $\mathbb{I}$:

$$\mathrm{d}\mu(y) \ = \ \bigotimes_{j\in\mathbb{Z}} \frac{1}{2} dy_j.$$

The sigma algebra Σ for μ is generated by the rectangles $\prod_{j\in\mathbb{N}} I_j$ where only a finite number of the I_j are different from $\mathbb{I}$. Then $(\mathbb{I}^\infty, \Sigma, \mu)$ is a probability space. Let $L_2(\mathbb{I}^\infty, \mu)$ denote the Hilbert space of functions on $\mathbb{I}^\infty$ equipped with the inner product

$$\langle f, g\rangle := \int_{\mathbb{I}^\infty} f(y)\overline{g(y)}\,\mathrm{d}\mu(y).$$

Consider two types of Legendre univariate polynomials expansions different only in their normalization for basis. The univariate Legendre basis $(P_n)_{n\in\mathbb{N}}$ is defined with $L_\infty(\mathbb{I})$-normalization: $\|P_n\|_{L_\infty(\mathbb{I})} = 1$. The orthonormal basis $(L_n)_{n\in\mathbb{Z}_+}$ in $L_2(\mathbb{I}, dy/2)$ for which $L_n = \sqrt{2n+1}\,P_n$ and $\|L_n\|_{L_2(\mathbb{I})} = 1$. Observe that $L_0 = P_0 = 1$ and there hold the Rodrigues formulas

$$P_n(t) \ = \ \frac{(-1)^n}{2^n n!}\,\frac{d^n}{dt^n}\Big[(1-t^2)^n\Big]. \tag{10}$$

Denote by $\mathbb{F}$ the subset in $\mathbb{Z}_+^\infty$ of all s such that $\mathrm{supp}(s)$ is finite, where $\mathrm{supp}(s)$ is the support of s, that is the set of all $j \in \mathbb{N}$ such that $s_j \ne 0$. We define the tensor products of these polynomials

$$P_s(y) := \prod_{j\in\mathbb{N}} P_{s_j}(y_j) \quad \text{and} \quad L_s(y) := \prod_{j\in\mathbb{N}} L_{s_j}(y_j), \quad s \in \mathbb{F}.$$

Then $(L_s)_{s\in\mathbb{F}}$ is an orthonormal basis of $L_2(\mathbb{I}^\infty, \mu)$.

Let X be a Banach space and $1 \le p \le \infty$. Denote by $L_\infty(\mathbb{I}^\infty, X)$ the space of all mappings v from $\mathbb{I}^\infty$ to X for which the following norm is finite

$$\|v\|_{L_\infty(\mathbb{I}^\infty, X)} := \sup_{y\in\mathbb{I}^\infty} \|v(y)\|_X.$$

We also use the notation

$$|v|_{L_\infty(\mathbb{I}^\infty, X)} := \sup_{y\in\mathbb{I}^\infty} |v(y)|_X$$

for a semi-norm $|v(y)|_X$ in X if any. he probability measure μ induces the Bochner space $L_p(\mathbb{I}^\infty, X, \mu)$ of μ-measurable mappings v from $\mathbb{I}^\infty$ to X which are p-summable. The norm in $L_p(\mathbb{I}^\infty, X, \mu)$ is defined by

$$\|v\|_{L_p(\mathbb{I}^\infty, X, \mu)} := \left(\int_{\mathbb{I}^\infty} \|v(\cdot, y)\|_X^p\,\mathrm{d}\mu(y)\right)^{1/p},$$

with the change to ess sup norm when $p = \infty$. For simplicity we identify $L_\infty(\mathbb{I}^\infty, X, \mu)$ with $L_\infty(\mathbb{I}^\infty, X)$. For a Hilbert space X and $p = 2$, the Bochner space $L_2(\mathbb{I}^\infty, X, \mu)$ coincides with the tensor product $X \otimes L_2(\mathbb{I}^\infty, \mu)$.

Due to (5) there hold the inclusions $u \in L_\infty(\mathbb{I}^\infty, V) \subset L_2(\mathbb{I}^\infty, V, \mu)$. Hence it follows that u admits the unique expansion

$$u \ = \ \sum_{s\in\mathbb{F}} u_s\,P_s \ = \ \sum_{s\in\mathbb{F}} v_s\,L_s, \tag{11}$$

converging in the Hilbert space $L_2(\mathbb{I}^\infty, V, \mu)$, where the Legendre coefficients u_s, v_s, are defined by

$$v_s := \langle u, L_s\rangle, \quad u_s := \prod_{j\in\mathbb{N}} (2s_j+1)^{1/2}\, v_s, \quad s \in \mathbb{F}. \tag{12}$$

Moreover, from the identity $L_2(\mathbb{I}^\infty, V, \mu) = V \otimes L_2(\mathbb{I}^\infty, \mu)$ it follows Parseval's identity

$$\|u\|^2_{L_2(\mathbb{I}^\infty, V, \mu)} \ = \ \sum_{s\in\mathbb{F}} \|v_s\|^2_V. \tag{13}$$

Similarly, assume that $a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$, then by (6) we have the inclusions $u \in L_\infty(\mathbb{I}^\infty, W) \subset L_2(\mathbb{I}^\infty, W, \mu)$, and therefore, the convergence of the Legendre expansion (11) in the Hilbert space $L_2(\mathbb{I}^\infty, W, \mu)$ and Parseval's identity

$$\|u\|^2_{L_2(\mathbb{I}^\infty, W, \mu)} \ = \ \sum_{s\in\mathbb{F}} \|v_s\|^2_W. \tag{14}$$

For $s \in \mathbb{F}$ with $\operatorname{supp}(s) \subset \{1, 2, ..., J\}$, we define the partial derivative

$$\partial^s_y u := \frac{\partial^{|s|} u}{\partial^{s_1} y_1 \cdots \partial^{s_J} y_J},$$

where $|s| := \sum_{j=1}^J |s_j|$.

It is known [10] that at any $y \in \mathbb{I}^\infty$, the function $y \mapsto u(y)$ admits a partial derivative $\partial^s_y u$. Moreover, starting with $u(y)$ which is the unique solution in V of the variational equation (9), we can recursively find all $\partial^s_y u(y)$ as the unique solution of the variational equation

$$\int_D a(y)(x) \nabla \partial^s_y u(y)(x) \cdot \nabla v(x)\, \mathrm{d}x \ = \ - \sum_{j:\ s_j \neq 0} s_j \int_D \psi_j(x)\, \nabla \partial^{s-e^j}_y u(y)(x) \cdot \nabla v(x)\, \mathrm{d}x. \quad \forall v \in V. \tag{15}$$

By use of (10) we derive from (15) by inductive integration by parts in the variables y_j the formulas for the Legendre coefficients

$$v_s \ = \ \frac{1}{s!} \prod_{j:\ s_j\neq 0} \frac{(2s_j+1)^{1/2}}{2^{s_j}} \int_{\mathbb{I}^\infty} \partial^s_y u(y) \prod_{j:\ s_j \neq 0} (1-y_j^2)^{s_j}\, \mathrm{d}\mu(y), \tag{16}$$

where $s! := \prod_{j=1}^J s_j!$.

Since $u \in L_2(\mathbb{I}^\infty, V, \mu)$, it can be defined as the unique solution of the variational problem: Find $u \in L_2(\mathbb{I}^\infty, V, \mu)$ such that

$$B(u, v) \ = \ F(v) \quad \forall v \in L_2(\mathbb{I}^\infty, V, \mu),$$

where

$$B(u, v) := \int_{\mathbb{I}^\infty} \int_D a(x, y) \nabla u(x, y) \cdot \nabla v(x, y)\, \mathrm{d}x\, \mathrm{d}\mu(y),$$

$$F(v) := \int_{\mathbb{I}^\infty} \int_D f(x)\, v(x, y)\, \mathrm{d}x\, \mathrm{d}\mu(y).$$

For a subset G in $\mathbb{Z}_+ \times \mathbb{F}$, denote by $\mathcal{V}(G)$ the subspace in $L_\infty(\mathbb{I}^\infty, V)$ of all functions v of the form

$$v(y)(x) \ = \ \sum_{(k,s)\in G} v_k(x)\, P_s(y), \quad y \in \mathbb{I}^\infty, \quad v_k \in V_{2^k},$$

and define the linear operator $\mathcal{S}_G : L_\infty(\mathbb{I}^\infty, V) \to \mathcal{V}(G)$ by

$$\mathcal{S}_G u(y)(x) := \sum_{(k,s)\in G} \delta_k(v_s)(x)\, L_s(y) \ = \ \sum_{(k,s)\in G} \delta_k(u_s)(x)\, P_s(y).$$

If G is a finite set, we define the *Galerkin approximation* u_G to u as the unique solution to the problem: Find $u_G \in \mathcal{V}(G)$ such that

$$B(u_G, v) \ = \ F(v) \quad \forall v \in \mathcal{V}(G).$$

By Céa's lemma we have the estimate

$$\|u - u_G\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ \sqrt{\frac{R}{r}} \inf_{v \in \mathcal{V}(G)} \|u - v\|_{L_2(\mathbb{I}^\infty,V,\mu)},$$

and consequently,

$$\|u - u_G\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ \sqrt{\frac{R}{r}} \, \|u - \mathcal{S}_G u\|_{L_2(\mathbb{I}^\infty,V,\mu)}. \tag{17}$$

For linear collective Galerkin approximations we need the following assumption.

Assumption (ii): There exist $0 < p < 2$, a positive sequence $\sigma = (\sigma_s)_{s \in \mathbb{F}}$ and a constant M such that the sequence $(\sigma_s^{-1})_{s \in \mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$ and

$$\|v_s\|_W \ \le M \, \sigma_s^{-1}, \quad s \in \mathbb{F}.$$

Let $0 < p < \infty$ and $\sigma := (\sigma_s)_{s \in \mathbb{F}}$ be a positive sequence. For $T > 0$, define the following subset in $\mathbb{Z}_+ \times \mathbb{F}$

$$G(T) \ = \ G_{p,\sigma}(T) := \ \left\{(k,s) \in \mathbb{Z}_+ \times \mathbb{F} : \, 2^k \sigma_s^p \le T\right\}. \tag{18}$$

Theorem 1. *Let Assumptions* (i) *and* (ii) *hold and* $a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$. *Then we have for every* $T > 0$,

$$\|u - u_{G(T)}\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ \sqrt{\frac{R}{r}} \left\|u - \mathcal{S}_{G(T)} u)\right\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ C \sqrt{\frac{R}{r}} \, T^{-\min(\alpha, 1/p - 1/2)},$$

where

$$C := \ M \, C_D \, \frac{2^\alpha + 1}{2^{\alpha^*} - 1} \left\|(\sigma_s^{-1})\right\|_{\ell_{p^*}(\mathbb{F})}^{p^*/2}$$

and $\alpha^* := \alpha$, $p^* := 2(1 - p\alpha)$ *for* $\alpha \le 1/p - 1/2$, *and* $\alpha^* := \alpha - 1/p + 1/2$, $p^* := p$ *for* $\alpha > 1/p - 1/2$.

Proof. We preliminarily show that

$$\lim_{N \to \infty} \|u - \mathcal{S}_{G_N}(u)\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ = \ 0, \tag{19}$$

where $G_N := \{(k,s) \in \mathbb{Z}_+ \times \mathbb{F} : \, 0 \le k \le N\}$. Obviously, by the definition,

$$\mathcal{S}_{G_N}(u) \ = \ \sum_{s \in \mathbb{F}} \sum_{k=0}^{N} \delta_k(v_s) \, L_s \ = \ \sum_{s \in \mathbb{F}} P_{2^N}(v_s) \, L_s.$$

By the assumptions we have the inclusion $u \in L_2(\mathbb{I}^\infty, W, \mu) \subset L_2(\mathbb{I}^\infty, V, \mu)$. From the uniform boundedness of the operators P_{2^N} and (13)

$$\|\mathcal{S}_{G_N}(u)\|^2_{L_2(\mathbb{I}^\infty,V,\mu)} \ = \ \sum_{s \in \mathbb{F}} \|P_{2^N}(v_s)\|_V^2 \ \le \ C_D^2 \sum_{s \in \mathbb{F}} \|v_s\|_V^2 \ = \ C_D^2 \, \|u\|^2_{L_2(\mathbb{I}^\infty,V,\mu)}.$$

This means that $\mathcal{S}_{G_N}(u) \in L_2(\mathbb{I}^\infty, V, \mu)$. Hence, by (13), Assumption (i) and (14) we deduce that

$$\|u - \mathcal{S}_{G_N}(u)\|^2_{L_2(\mathbb{I}^\infty,V,\mu)} \ = \ \sum_{s \in \mathbb{F}} \|v_s - P_{2^N}(v_s)\|_V^2 \ \le \ C_D^2 \, 2^{-2\alpha N} \sum_{s \in \mathbb{F}} \|v_s\|_W^2 \ = \ C_D^2 \, 2^{-2\alpha N} \|u\|^2_{L_2(\mathbb{I}^\infty,W,\mu)}$$

which prove (19).

Let T be given and ε arbitrary positive number. Then since $G(T)$ is finite from the definition of G_N and (19) there exists $N = N(T, \varepsilon)$ such that $G(T) \subset G_N$ and

$$\|u - \mathcal{S}_{G_N}(u)\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ \varepsilon. \tag{20}$$

By the triangle inequality,

$$\|u - \mathcal{S}_{G(T)} u\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ \|u - \mathcal{S}_{G_N}(u)\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ + \ \|\mathcal{S}_{G_N}(u) - \mathcal{S}_{G(T)} u\|_{L_2(\mathbb{I}^\infty,V,\mu)}. \tag{21}$$

We prove that

$$\|\mathcal{S}_{G_N}(u) - \mathcal{S}_{G(T)} u\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le \ C \, T^{-\min(\alpha, 1/p - 1/2)}. \tag{22}$$

Let us first consider the case $\alpha \le 1/p - 1/2$. We have by (13) and (8) that

$$\begin{aligned}
\|\mathcal{S}_{G_N}(u) - \mathcal{S}_{G(T)}u\|^2_{L_2(\mathbb{I}^\infty,V,\mu)} &= \Big\|\sum_{s\in\mathbb{F}}\sum_{k=0}^{N}\delta_k(v_s)\,L_s - \sum_{s\in\mathbb{F}}\sum_{2^k>T\sigma_s^{-p}}\delta_k(v_s)\,L_s\Big\|^2_{L_2(\mathbb{I}^\infty,V)} \\
&= \Big\|\sum_{s\in\mathbb{F}}\sum_{T\sigma_s^{-p}<2^k<N}\delta_k(v_s)\,L_s\Big\|^2_{L_2(\mathbb{I}^\infty,V)} \\
&= \sum_{s\in\mathbb{F}}\Big\|\sum_{T\sigma_s^{-p}<2^k<N}\delta_k(v_s)\Big\|^2_V \\
&\le \sum_{s\in\mathbb{F}}\Big(\sum_{T\sigma_s^{-p}<2^k<N}\|\delta_k(v_s)\|_V\Big)^2 \\
&\le \sum_{s\in\mathbb{F}}\Big(\sum_{T\sigma_s^{-p}<2^k<N}(2^\alpha+1)C_D\,2^{-\alpha k}\|v_s\|_W\Big)^2 \\
&\le (2^\alpha+1)^2C_D^2\sum_{s\in\mathbb{F}}\|v_s\|^2_W\Big(\sum_{2^k>T\sigma_s^{-p}}2^{-\alpha k}\Big)^2.
\end{aligned}$$

Hence, by Assumption (ii) and the inequality $2(1-p\alpha)\ge p$ we derive that

$$\begin{aligned}
\|\mathcal{S}_{G_N}(u) - \mathcal{S}_{G(T)}u\|^2_{L_2(\mathbb{I}^\infty,V,\mu)} &\le (2^\alpha+1)^2C_D^2\sum_{s\in\mathbb{F}}\sigma_s^{-2}\Big(\sum_{2^k>T\sigma_s^{-p}}2^{-\alpha k}\Big)^2 \\
&\le T^{-2\alpha}M^2C_D^2\frac{(2^\alpha+1)^2}{(2^\alpha-1)^2}\sum_{s\in\mathbb{F}}\sigma_s^{-2(1-p\alpha)} \\
&= C^2\,T^{-2\alpha}.
\end{aligned}$$

We next consider the case $\alpha > 1/p - 1/2$. Again, we have by (13), (8) and Assumption (ii) that

$$\begin{aligned}
\|\mathcal{S}_{G_N}(u) - \mathcal{S}_{G(T)}u\|_{L_2(\mathbb{I}^\infty,V,\mu)} &= \Big\|\sum_{k=0}^{N}\sum_{s\in\mathbb{F}}\delta_k(v_s)\,L_s - \sum_{k=0}^{N}\sum_{\sigma_s\le(T2^{-k})^{1/p}}\delta_k(v_s)\,L_s\Big\|_{L_2(\mathbb{I}^\infty,V)} \\
&\le \sum_{k=0}^{N}\Big\|\sum_{\sigma_s>(T2^{-k})^{1/p}}\delta_k(v_s)\,L_s\Big\|_{L_2(\mathbb{I}^\infty,V)} \\
&= \sum_{k=0}^{N}\Big(\sum_{\sigma_s^{-1}<(T2^{-k})^{-1/p}}\|\delta_k(v_s)\|^2_V\Big)^{1/2} \\
&\le \sum_{k=0}^{N}\Big(\sum_{\sigma_s^{-1}<(T2^{-k})^{-1/p}}\Big((2^\alpha+1)C_D\,2^{-\alpha k}\|v_s\|_W\Big)^2\Big)^{1/2} \\
&\le (2^\alpha+1)C_DM\sum_{k=0}^{N}2^{-\alpha k}\Big(\sum_{\sigma_s^{-1}<(T2^{-k})^{-1/p}}\sigma_s^{-2}\Big)^{1/2} \\
&= (2^\alpha+1)C_DM\sum_{k=0}^{N}2^{-\alpha k}\Big(\sum_{\sigma_s^{-1}<(T2^{-k})^{-1/p}}\sigma_s^{-2+p}\sigma_s^{-p}\Big)^{1/2} \\
&\le (2^\alpha+1)C_DM\,T^{-(1/p-1/2)}\sum_{k=0}^{\infty}2^{-(\alpha-1/p+1/2)k}\Big(\sum_{s\in\mathbb{F}}\sigma_s^{-p}\Big)^{1/2} \\
&= C\,T^{-(1/p-1/2)}.
\end{aligned}$$

The estimate (22) is proven. This estimate in combining with (20) and (21) gives

$$\|u - \mathcal{S}_{G(T)}u\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le\ \varepsilon \ +\ C\,T^{-\min(\alpha,1/p-1/2)}$$

for arbitrary positive number ε. Hence,

$$\|u - \mathcal{S}_{G(T)}u\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le\ C\,T^{-\min(\alpha,1/p-1/2)}$$

which together with (17) proves the theorem. ∎

We show that under the assumptions of Theorem 1, for a given $n \in \mathbb{N}$, the respective operator $\mathcal{S}_{G(T_n)}$ with properly chosen $p = \frac{2}{1+2\alpha}$ and $T = T_n$ is a bounded linear operator in $L_\infty(\mathbb{I}^\infty, V)$ of rank $\le n$ which gives the convergence rate of the approximation to $u(y)$ as $n^{-\alpha}$. For any $n \in \mathbb{N}$, let T_n be the number defined by the inequalities

$$2\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^p\,T_n \ \le\ n \ <\ 4\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^p\,T_n. \tag{23}$$

Theorem 2. *Let the assumptions and notation of Theorem 1 hold. For any* $n \in \mathbb{N}$, *let* T_n *be the number defined as in* (23) *and put* $\mathcal{V}_n := \mathcal{V}\big(G(T_n)\big)$, $\mathcal{P}_n := \mathcal{S}_{G(T_n)}$, $u_n := u_{G(T_n)}$. *Then*

- $\big\{\mathcal{V}_n\big\}_{n\in\mathbb{Z}_+}$ *is a nested sequence of subspaces in* $L_2(\mathbb{I}^\infty, V, \mu)$ *and* $\dim \mathcal{V}_n \le n$;
- $\big\{\mathcal{P}_n\big\}_{n\in\mathbb{Z}_+}$ *is a sequence of linear bounded operators from* $L_2(\mathbb{I}^\infty, V, \mu)$ *into* $\mathcal{V}_n$; *and*
- *for every* $n \in \mathbb{N}$,

$$\|u - u_n\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le\ \sqrt{\frac{R}{r}}\,\big\|u - \mathcal{P}_n u\big\|_{L_2(\mathbb{I}^\infty,V,\mu)} \ \le\ C'\sqrt{\frac{R}{r}}\,n^{-\min(\alpha,1/p-1/2)},$$

where

$$C' := C4^\alpha\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^{p\alpha}.$$

In particular, if in addition, $p = \frac{2}{1+2\alpha}$ *in Assumption* (ii), *then we have that*

$$\big\|u - \mathcal{P}_n u\big\|_{L_2(\mathbb{I}^\infty,V)} \ \le\ C'\,n^{-\alpha}, \tag{24}$$

Proof. We have that

$$\begin{aligned}
\dim \mathcal{V}(G(T)) \ &\le\ \sum_{(k,s)\in G(T)} \dim V_{2^k} \ \le\ \sum_{(k,s)\in G(T)} 2^k \\
&\le\ \sum_{\sigma_s^p\ \le\ T}\ \sum_{2^k\ \le T\sigma_s^{-p}} 2^k \ \le\ 2\sum_{\sigma_s^p\ \le\ T} T\sigma_s^{-p} \\
&\le\ 2\,T\sum_{s\in\mathbb{F}}\sigma_s^{-p} \ \le\ 2\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^p\,T.
\end{aligned}$$

Hence, by (23) we derive that

$$\dim \mathcal{V}(G(T_n)) \ \le\ 2\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^p\,T_n \ \le\ n. \tag{25}$$

On the other hand, by (23),

$$T_n^{-\alpha} \ \le\ 4^\alpha\big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^{p\alpha}\,n^{-\alpha}$$

which together with Theorem 1 and (25) completes the proof of the theorem. ∎

From Theorem 1 we see that the problem of construction of a linear collective Galerkin approximation is reduced to the construction of a sequence $\sigma = (\sigma_s)_{s\in\mathbb{F}}$ satisfying Assumption (ii). There may be several ways to construct such a sequence. Here, we present a way based an estimate for $\|\partial_y^s u\|_{L_\infty(\mathbb{I}^\infty,W)}$. We define the following constant K and sequence b as follows.

$$K := \ \frac{1}{r}\left[1 + \left(1 + \frac{|a|_{L_\infty(\mathbb{I}^\infty,W^1_\infty(D))}}{r}\right)\right]\|f\|_{L_2(D)}; \tag{26}$$

$$b \ =\ (b_j)_{j\in\mathbb{N}}, \quad b_j := \ \frac{1}{r}\left(\left(\frac{|a|_{L_\infty(\mathbb{I}^\infty,W^1_\infty(D))}}{r} + 2\right)\|\psi_j\|_{L_\infty(D)} + |\psi_j|_{W^1_\infty(D)}\right). \tag{27}$$

See [12, Lemma 3.4] for a proof of the following lemma.

Lemma 1. *Assume that* $a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$. *Then we have*

$$\|\partial_y^s u\|_{L_\infty(\mathbb{I}^\infty,W)} \ \le \ K|s|!\, b^s, \quad s \in \mathbb{F}.$$

Lemma 2. *Assume that* $a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$. *Define the sequence*

$$d = (d_j)_{j\in\mathbb{N}}, \quad d_j := b_j/\sqrt{3}. \tag{28}$$

Then we have

$$\|v_s\|_W \ \le \ K\frac{|s|!}{s!}\, d^s, \quad s \in \mathbb{F}.$$

Proof. From (16) we derive that

$$\|v_s\|_W \ \le \ \frac{3^{-|s|/2}}{s!}\, \|\partial_y^s u\|_{L_\infty(\mathbb{I}^\infty,W)}$$

which combining with Lemma 1 proves the lemma. □

Lemma 3. *Let* $0 < p < \infty$, $c = (c_j)_{j\in\mathbb{N}}$ *be a positive sequence. Then we have the following.*

$$\left(\frac{|s|!}{s!}\, c^s\right) \in \ell_p(\mathbb{F}) \iff \begin{cases} \|c\|_{\ell_1(\mathbb{N})} \ < \ 1, \ c \in \ell_p(\mathbb{N}), & \textit{for } p \le 1; \\ \|c\|_{\ell_1(\mathbb{N})} \ \le \ 1, & \textit{for } p > 1. \end{cases}$$

This lemma was proven in [10, Theorem 7.2] for $p \le 1$ and in [14, Theorem 5.2] for $p > 1$. From Lemmata 2 and 3 we obtain the following corollary.

Corollary 1. Let the function a belong to $L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$, $p(\alpha) = \frac{2}{1+2\alpha}$ and the sequence $d = (d_j)_{j\in\mathbb{N}}$ defined in (28) satisfy the condition

$$\begin{cases} \|d\|_{\ell_1(\mathbb{N})} \ < \ 1, \ d \in \ell_p(\mathbb{N}), & \text{for } p \le 1; \\ \|d\|_{\ell_1(\mathbb{N})} \ \le \ 1, & \text{for } p > 1. \end{cases}$$

Then there holds Assumption (ii) for $M = K$ and the sequence

$$\sigma := (\sigma_s)_{s\in\mathbb{F}}, \quad \sigma_s^{-1} := \frac{|s|!}{s!} d^s.$$

3. Legendre approximation

The collective Legendre approximation is constructed on the basis of a representation of the solution u by a series converging unconditionally in $L_\infty(\mathbb{I}^\infty, V)$ as in the following lemma.

We say that a sequence $(\Lambda_N)_{N\in\mathbb{N}} \subset \mathbb{F}$ of finite sets exhausts $\mathbb{F}$ if any finite set $\Lambda \subset \mathbb{F}$ is contained in all Λ_N for $N \ge N_0$ with N_0 sufficiently large. Similarly, we say that a sequence $(G_N)_{N\in\mathbb{N}} \subset \mathbb{Z}_+ \times \mathbb{F}$ of finite sets exhausts $\mathbb{Z}_+ \times \mathbb{F}$ if any finite set $G \subset \mathbb{Z}_+ \times \mathbb{F}$ is contained in all G_N for $N \ge N_0$ with N_0 sufficiently large.

Lemma 4. *Let Assumption* (i) *hold and let the sequence* $\left(\|\psi_j\|_{W^1_\infty(D)}\right)_{j\in\mathbb{N}}$ *belong to* $\in \ell_1(\mathbb{N})$. *Then* $(\|u_s\|_W)_{s\in\mathbb{F}}$ *belongs to* $\ell_1(\mathbb{F})$ *and* $u(y)$ *can be represented as the series*

$$u(y) \ = \sum_{(k,s)\in\mathbb{Z}_+\times\mathbb{F}} \delta_k(u_s)\, P_s(y), \quad y \in \mathbb{I}^\infty, \tag{29}$$

converging unconditionally in $L_\infty(\mathbb{I}^\infty, V)$.

Proof. To prove the lemma we need the following fact. If the sequence $\left(\|\psi_j\|_{L_\infty(D)}\right)_{j\in\mathbb{N}}$ belong to $\ell_1(\mathbb{N})$, then $u_s \in V$ belongs to $\ell_1(\mathbb{F})$ and the Legendre expansion (11) converges unconditional in $L_\infty(\mathbb{I}^\infty, V)$. This assertion can be proven in a similar way to the proof of [11, Theorem 4.1] which states that if $\left(\|\psi_j\|_{L_\infty(D)}\right)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ for some $0 < p < 1$, then $(\|u_s\|_V)_{s\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$, and there hold the expansions (11) with unconditional convergence in $L_\infty(\mathbb{I}^\infty, V)$.

Let us prove the lemma. The inclusion $(\|u_s\|_W)_{s\in\mathbb{F}} \in \ell_1(\mathbb{F})$ can be proven in a similar way to the proof of [11, Theorem 5.1] which states that if $\left(\|\psi_j\|_{W^1_\infty(D)}\right)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$ for some $0 < p < 1$, then $(\|t_s\|_W)_{s\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$. We now prove the unconditional convergence of the series (29).

It can be proven in a way similar to the proof of [12, Lemma 2.1]. For completeness let us process a detailed proof. We first prove the convergence of the series (29) for a sequence of special form $(G_N^*)_{N\in\mathbb{N}}$ with

$$G_N^* = \{(k,s) \in \mathbb{Z}_+ \times \mathbb{F} : 0 \le k \le N,\ s \in \Lambda_N\},$$

for any sequence $(\Lambda_N)_{N\in\mathbb{N}}$ is of finite subsets in $\mathbb{F}$ which exhausts $\mathbb{F}$. We have for every $y \in \mathbb{I}^\infty$ that

$$\Big\|u(y) - \sum_{(k,s)\in G_N^*} \delta_k(u_s) P_s(y)\Big\|_V \le \Big\|u(y) - \sum_{s\in\Lambda_N} u_s P_s(y)\Big\|_V + \Big\|\sum_{s\in\Lambda_N} u_s P_s(y) - \sum_{(k,s)\in G_N^*} \delta_k(u_s) P_s(y)\Big\|_V.$$

Since the Legendre expansions (11) converge unconditional in $L_\infty(\mathbb{I}^\infty, V)$, it is enough to prove that

$$\lim_{N\to\infty} \Big\|\sum_{s\in\Lambda_N} u_s P_s - \sum_{(k,s)\in G_N^*} \delta_k(u_s) P_s\Big\|_{L_\infty(\mathbb{I}^\infty,V)} = 0. \tag{30}$$

By Assumption (i) we get that for every $y \in \mathbb{I}^\infty$,

$$\begin{aligned}
\Big\|\sum_{s\in\Lambda_N} u_s P_s(y) - \sum_{(k,s)\in G_N^*} \delta_k(u_s) P_s(y)\Big\|_V &= \Big\|\sum_{s\in\Lambda_N} u_s P_s(y) - \sum_{s\in\Lambda_N}\sum_{k=0}^{N} \delta_k(u_s) P_s(y)\Big\|_V \\
&= \Big\|\sum_{s\in\Lambda_N} \Big[u_s - P_{2^N}(u_s)\Big] P_s(y)\Big\|_V \le \sum_{s\in\Lambda_N} \|u_s - P_{2^N}(u_s)\|_V \\
&\le \sum_{s\in\Lambda_N} C_D\, 2^{-\alpha N}\|u_s\|_W \le C_D\, 2^{-\alpha N}\|(\|u_s\|_W)\|_{\ell_1(\mathbb{F})}
\end{aligned}$$

which proves (30).

Take any sequence $(G_N)_{N\in\mathbb{N}}$ of finite subsets in $\mathbb{Z}_+ \times \mathbb{F}$ which exhausts $\mathbb{Z}_+ \times \mathbb{F}$. Then for arbitrary number $\varepsilon > 0$, there exists $M = M(\varepsilon)$ such that

$$\Big\|u - \sum_{(k,s)\in G_M^*} \delta_k(u_s) P_s\Big\|_{L_\infty(\mathbb{I}^\infty,V)} \le \frac{\varepsilon}{2}.$$

We obtain by (8) that

$$\sum_{(k,s)\notin G_M^*} \|\delta_k(u_s)\|_V \le (2^\alpha + 1) C_D\, 2^{-\alpha M} \sum_{(k,s)\notin G_M^*} \|u_s\|_W \le (2^\alpha + 1) C_D\, 2^{-\alpha M} \|(\|u_s\|_W)\|_{\ell_1(\mathbb{F})}.$$

Therefore, we may also assume that

$$\sum_{(k,s)\notin G_M^*} \|\delta_k(u_s)\|_V \le \frac{\varepsilon}{2}.$$

Since $(G_N)_{N\in\mathbb{N}}$ exhausts $\mathbb{Z}_+ \times \mathbb{F}$, there exists N^* such that $G_M^* \subset G_N$ for all $N \ge N^*$. Hence we derive that for arbitrary number $\varepsilon > 0$,

$$\Big\|u - \sum_{(k,s)\in G_N^*} \delta_k(u_s) P_s\Big\|_{L_\infty(\mathbb{I}^\infty,V)} \le \Big\|u - \sum_{(k,s)\in G_M^*} \delta_k(u_s) P_s\Big\|_{L_\infty(\mathbb{I}^\infty,V)} + \sum_{(k,s)\notin G_M^*} \|\delta_k(u_s)\|_V \le \varepsilon.$$

This proves the lemma. ∎

For the linear collective Legendre approximation to the solution u we need the following assumption.

Assumption (iii): There exist $0 < p < 1$, a positive sequence $\sigma = (\sigma_s)_{s\in\mathbb{F}}$ and a constant M such that the sequence $(\sigma_s^{-1})_{s\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$ and

$$\|u_s\|_W \le M\, \sigma_s^{-1}, \quad s \in \mathbb{F}.$$

Theorem 3. *Let Assumptions* (i) *and* (iii) *hold. For* $T > 0$, *consider the set* $G(T) = G_{p,\sigma}(T)$ *as in* (18). *Then we have for every* $T > 0$,

$$\Big\|u - \mathcal{S}_{G(T)} u\Big\|_{L_\infty(\mathbb{I}^\infty,V)} \le C\, T^{-\min(\alpha, 1/p-1)},$$

where

$$C := \; M\, C_D\, \frac{2^\alpha+1}{2^{\alpha^*}-1}\, \big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_{p^*}(\mathbb{F})}^{p^*},$$

and $\alpha^* := \alpha$, $p^* := 1 - p\alpha$ *for* $\alpha \le 1/p - 1$, *and* $\alpha^* := \alpha - 1/p + 1$, $p^* := p$ *for* $\alpha > 1/p - 1$.

Proof. By Lemma 4 the series (29) converging unconditionally in $L_\infty(\mathbb{I}^\infty, V)$ to u , and, consequently, we can write for every $y \in \mathbb{I}^\infty$,

$$\Big\|u(y) - \mathcal{S}_{G(T)}u(y)\Big\|_V \;=\; \Big\|\sum_{(k,s)\notin G(T)} \delta_k(u_s)\, P_s(y)\Big\|_V \;\le\; \sum_{(k,s)\notin G(T)} \|\delta_k(u_s)\|_V.$$

By (8) we have that

$$\|\delta_k(u_s)\|_V \;\le\; (2^\alpha+1)\, C_D\; 2^{-\alpha k}\|u_s\|_W \quad \forall (k,s) \in \mathbb{Z}_+ \times \mathbb{F}.$$

Therefore, by Assumption (iii)

$$\Big\|u - \mathcal{S}_{G(T)}u\Big\|_{L_\infty(\mathbb{I}^\infty, V)} \;\le\; (2^\alpha+1) C_D\, M \sum_{(k,s)\notin G(T)} 2^{-\alpha k}\sigma_s^{-1}.$$

Hence and from the inequality [12, Lemma 2.2]

$$\sum_{(k,s)\notin G(T)} 2^{-\alpha k}\sigma_s^{-1} \;\le\; C^*\, T^{-\min(\alpha, 1/p-1)} \tag{31}$$

with

$$C^* := \; \frac{1}{2^{\alpha^*}-1}\, \big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_{p^*}(\mathbb{F})}^{p^*} T^{-\min(\alpha,1/p-1)},$$

we obtain the theorem. For completeness, let us prove (31). If $\alpha \le 1/p - 1$, we have for every $N \in \mathbb{N}$,

$$\sum_{(k,s)\notin G(T)} 2^{-\alpha k}\sigma_s^{-1} \;\le\; \sum_{s\in\mathbb{F}} \sigma_s^{-1} \sum_{2^k \,>\, T\sigma_s^{-p}} 2^{-\alpha k} \;\le\; \sum_{s\in\mathbb{F}} \frac{1}{2^\alpha-1}\sigma_s^{-1}\Big(T\sigma_s^{-p}\Big)^{-\alpha}$$

$$= \; \frac{T^{-\alpha}}{2^\alpha-1} \sum_{s\in\mathbb{F}} \sigma_s^{-(1-p\alpha)} \;\le\; C^*\, T^{-\alpha}.$$

If $\alpha > 1/p - 1$, we have for every $N \in \mathbb{N}$,

$$\sum_{(k,s)\notin G(T)} 2^{-\alpha k}\sigma_s^{-1} \;\le\; \sum_{k\,\ge 0} 2^{-\alpha k} \sum_{\sigma_s \ge (T2^{-k})^{1/p}} \sigma_s^{-1} \;\le\; \sum_{k\,\ge 0} 2^{-\alpha k} \sum_{\sigma_s \ge (T2^{-k})^{1/p}} \sigma_s^{-(1-p)}\sigma_s^{-p}$$

$$= \; T^{-(1/p-1)} \sum_{k\,\ge 0} 2^{-(\alpha-1/p+1)k} \sum_{\sigma_s\in\mathbb{F}} \sigma_s^{-p} \;\le\; C^*\, T^{-(1/p-1)}.$$

In the last step we used the inequality $\alpha - 1/p + 1 > 0$. The inequality (31) is proven. ∎

Similarly to the proof of Theorem 2, from Theorem 3 we derive the following

Theorem 4. *Let the assumptions and notation of Theorem 3 hold. For any* $n \in \mathbb{N}$, *let* T_n *be the number defined as in* (23) *and put* $\mathcal{V}_n := \mathcal{V}\big(G(T_n)\big)$, $\mathcal{P}_n := \mathcal{S}_{G(T_n)}$. *Then*

- $\big\{\mathcal{V}_n\big\}_{n\in\mathbb{Z}_+}$ *is a nested sequence of subspaces in* $L_\infty(\mathbb{I}^\infty, V)$ *and* $\dim \mathcal{V}_n \le n$ *;*
- $\big\{\mathcal{P}_n\big\}_{n\in\mathbb{Z}_+}$ *is a sequence of linear bounded operators from* $L_\infty(\mathbb{I}^\infty, V)$ *into* $\mathcal{V}_n$ *; and*
- *for every* $n \in \mathbb{N}$ *,*

$$\big\|u - \mathcal{P}_n u\big\|_{L_\infty(\mathbb{I}^\infty, V)} \;\le\; C'\, n^{-\min(\alpha, 1/p-1)},$$

where

$$C' := C\, 4^\alpha\, \big\|\big(\sigma_s^{-1}\big)\big\|_{\ell_p(\mathbb{F})}^{p\alpha}.$$

Moreover, if in addition, $p = \frac{1}{1+\alpha}$ *in Assumption* (ii) *, then we have that*

$$\big\|u - \mathcal{P}_n u\big\|_{L_\infty(\mathbb{I}^\infty, V)} \;\le\; C'\, n^{-\alpha}.$$

From Theorems 3 and 4 we see that the problem of construction of a linear collective Legendre approximation is reduced to the construction of a sequence $\sigma = (\sigma_s)_{s\in\mathbb{F}}$ satisfying Assumption (iii).

Corollary 2. Let the constant K be as in (26) and the sequence b as in (27). Assume that the function $a \in L_\infty(\mathbb{I}^\infty, W^1_\infty(D))$, there exists $0 < p < 1$ such that the sequence $\big(\|\psi_j\|_{W^1_\infty(D)}\big)_{j\in\mathbb{N}}$ belongs to $\ell_p(\mathbb{N})$ and $\|b\|_{\ell_1(\mathbb{N})} < 1$. Then there holds Assumption (iii) for p, $M = K$ and the sequence

$$\sigma := \big(\sigma_s\big)_{s\in\mathbb{F}}, \quad \sigma_s^{-1} := \frac{|s|!}{s!} b^s.$$

Proof. By using of (12), (16) and Lemma 1 we derive that

$$\|u_s\|_W \ \le\ \frac{1}{s!}\, \|\partial_y^s u\|_{L_\infty(\mathbb{I}^\infty, W)} \ \le\ K\frac{|s|!}{s!} b^s \ =\ K\,\sigma_s^{-1}.$$

On the other hand, from the assumptions we have that $b \in \ell_p(\mathbb{N})$ and $\|b\|_{\ell_1(\mathbb{N})} < 1$. Hence by Lemma 3 the sequence $(\sigma_s^{-1})_{s\in\mathbb{F}}$ belongs to $\ell_p(\mathbb{F})$. This proves the corollary. ∎

Acknowledgments. This work is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 102.01-2017.05. A part of this work was done when the author was working as a research professor at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the VIASM for providing a fruitful research environment and working condition. He expresses a special thank to Christoph Schwab for valuable remarks and suggestions.

References

1 Andreev R., Bieri M., Schwab Ch. Sparse tensor discretization of elliptic sPDEs// SIAM J. Sci. Comput. -2009. **31**/ -P. 4281–4304 .

2 Babuska I., Tempone R., Zouraris G.E. Galerkin finite element approximations of stochastic elliptic partial differential equations// SIAM J. Numer. Anal. -2004. **42**. -P. 800–825.

3 Bachmayr M., Cohen A., Migliorati G. Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients, ESAIM: M2AN/ -2017. **51**. -P. 321–339.

4 Bachmayr M., Cohen A., Dũng D., C. Schwab Fully discrete approximation of parametric and stochastic elliptic PDEs// SIAM J. Numer. Anal. -2017. **55**. -P. 2151– 2186.

5 Bäck J., Nobile F., Tamellini L., Tempone R. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods// Math. Mod. Methods Appl. Sci. -2012. **22**. -P. 1250023.

6 Ciarlet P. The finite element method for elliptic problems, North Holland Publ., 1978.

7 Chkifa A., Cohen A., DeVore R., Schwab Ch. Sparse adaptive Taylor approximation algorithms forparametric and stochastic elliptic PDEs// Modl. Math. Anal. Numr. -2013. **47**(1). -P. 253–280.

8 Chkifa A., Cohen A., Schwab C. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs// Found. Comp. Math -2014. **14**. -P. 601–633.

9 Cohen A., DeVore R. Approximation of high-dimensional parametric PDEs// Acta Numerica -2015. **24**. -P. 1–159.

10 Cohen A., DeVore R., Schwab Ch. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs// Found. Comput. Math. -2010. **10**. -P. 615–646.

11 Cohen A., DeVore R., Schwab Ch. Analytic regularity and polynomial approximation of parametric and stochastic PDEs// Anal. Appl. -2010. **9**. -P. 11–47.

12 Dũng D. Linear collective collocation and Galerkin approximations for parametric and stochastic elliptic PDEs, arXiv:1511.03377, 2015.

13 Dũng D., Griebel M. Hyperbolic cross approximation in infinite dimensions//J. Complexity -2016. **33**. -P. 33–88.

14 Dũng D., Griebel M., Vu N.H., Rieger C. ε-dimension in infinite dimensional hyperbolic cross approximation and application to parametric elliptic PDEs// J. Complexity -2018. 46. -P. 66-89.

15 Dũng D., Temlyakov V., Ullrich T. Hyperbolic Cross Approximation, Series "Advanced Courses in Mathematics, CRM Barcelona", Birkhäuser, Basel, 2018.

16 Frauenfelder Ph., Schwab Ch., Todor R.A. Finite elements for elliptic problems with stochastic coefficients// Comput. Methods Appl. Mech. Eng. -2005. **194**. -P. 205–228 .

17 Gittelson C.J. An adaptive stochastic Galerkin method for random elliptic operators// Math. Comp. -2013. **82**. -P. 1515–1541.

18 Gunzburger M., Webster C., Zang G. Stochastic finite element methods for partial diferential equations with random input data// Acta Numerica -2014. **23**. -P. 521–650.

19 Hoang V.H., Schwab C. N-term Galerkin Wiener chaos approximation rates for elliptic PDEs with lognormal Gaussian random inputs// Math. Models Methods Appl. Sci. -2014. **24**. -P. 797–826.

20 Matthies H.G., Keese A. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Engrg. -2005. **194**. -P. 1295–1331.
21 Nistor V., Schwab Ch. High order Galerkin approximations for parametric, second order ellipticpartial differential equations// Math. Models Methods Appl. Sci. -2013. **123**. -P. 1729–1760.
22 Schwab Ch., Gittelson C. Sparse tensor discretizations high-dimensional parametric and stochastic PDEs//Acta Numerica -2011. **120**. -P. 291–467.
23 Zech J., Dũng D., Schwab Ch., Multilevel approximation of parametric and stochastic PDEs, Research Report No. 2018-05, Seminar for Appied Mathematics, ETH.

Динь Зунг

Информациялық технологиялар институты, Вьетнам ұлттық университеті, Суан Тхюи 144, Каузяй, Ханой, Вьетнам

Параметрлі және стохастикалық эллиптикалық дербес туындылы теңдеулердің шешімдерінің Галеркин жуықтаулары

Аннотация: Шенелген Липшиц облысы $D \subset \mathbb{R}^m$, $\mathbb{I}^\infty := [-1,1]^\infty$, $f \in L_2(D)$, бірқалыпты эллиптикалық шартты қанағаттандыратын және y-қа аффинді тәуелді $a(y)$ диффузиясы жағдайларындағы

$$-\operatorname{div}\big(a(y)(x)\nabla u(y)(x)\big) = f(x) \quad x \in D,\ y \in \mathbb{I}^\infty, \quad u|_{\partial D} = 0,$$

параметрлік эллиптикалық теңдеу үшін Галеркиннің жуықтауы зерттеледі. Параметрлік емес $-\operatorname{div}\big(a(y_0)(x)\nabla u(y_0)(x)\big) = f(x)$ теңдеуі үшін $\mathbb{I}^\infty$ жиынының y_0 барлық нүктелерінде дерлік $\mathbb{I}^\infty$ жиынындағы μ бірқалыпты ықтималдық өлшемі бойынша $V := H_0^1(D)$ кеңістігінің энергетикалық нормасында шешіміне қайсібір жылдамдықпен жинақталатын ақырлы элементтер тізбегі табылады делік. Осы болжамға сүйене отырып, $L_2(\mathbb{I}^\infty, V, \mu)$ Бохнер кеңістіктерінің нормасында параметрлік эллиптикалық мәселе үшін сол жылдамдықпен шешіміне ұмтылатын ақырлы элементтер тізбегі құрылды. Бұл параметрлік эллиптикалық мәселе үшін өлшемділіктің қиянаттының қасіретін сызықтық әдістер арқылы жойылатыны көрсетілді.

Түйін сөздер: Параметрлі және стохастикалық эллиптикалық дербес туындылы теңдеулер, Галеркиннің бірлескен жуықтауы, диффузия коэффициенттерінің аффиндық тәуелділігі, өлшемділіктің қиянатты қасіреті.

Динь Зунг

Институт информационных технологий, Вьетнамский национальный университет, Суан Тхюи 144, Каузяй, Ханой, Вьетнам

Приближения Галеркина решений параметрических и стохастических эллиптических уравнений в частных производных

Аннотация: Изучается приближение Галеркина для параметрической эллиптической задачи

$$-\operatorname{div}\big(a(y)(x)\nabla u(y)(x)\big) = f(x) \quad x \in D,\ y \in \mathbb{I}^\infty, \quad u|_{\partial D} = 0,$$

где $D \subset \mathbb{R}^m$ - ограниченная область Липшица, $\mathbb{I}^\infty := [-1,1]^\infty$, $f \in L_2(D)$, а диффузия $a(y)$ удовлетворяет условию равномерной эллиптичности и аффинно зависит от y. Предположим, что почти в каждой точке $y_0 \in \mathbb{I}^\infty$ относительно равномерной вероятностной меры μ на $\mathbb{I}^\infty$ для непараметрической задачи $-\operatorname{div}\big(a(y_0)(x)\nabla u(y_0)(x)\big) = f(x)$ существует аппроксимативная последовательность конечных элементов с определенной скоростью сходимости по энергетической норме пространства $V := H_0^1(D)$. На основе этого предположения построена последовательность конечных элементов с той же скоростью сходимости для параметрической эллиптической задачи по норме $L_2(\mathbb{I}^\infty, V, \mu)$ пространств Бохнера. Это показывает, что проклятия размерности для параметрической эллиптической задачи преодолевается линейными методами.

Ключевые слова: параметрические и стохастические эллиптические уравнения в частных производных, совместное приближение Галеркина, аффинная зависимость коэффициентов диффузии, проклятие размерности.

References

1 R. Andreev, M. Bieri and Ch. Schwab, Sparse tensor discretization of elliptic sPDEs, SIAM J. Sci. Comput. **31**(2009), 4281–4304 .
2 I. Babuska, R. Tempone and G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations, SIAM J. Numer. Anal. **42** (2004), 800–825.

3 M. Bachmayr, A. Cohen and G. Migliorati, Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients, ESAIM: M2AN **51**(2017), 321–339.
4 M. Bachmayr, A. Cohen, D. Dũng and C. Schwab, Fully discrete approximation of parametric and stochastic elliptic PDEs, SIAM J. Numer. Anal. **55**(2017), 2151– 2186.
5 J. Bäck, F. Nobile, L. Tamellini and R. Tempone, On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods, Math. Mod. Methods Appl. Sci. **22**(2012), 1250023.
6 P. Ciarlet, The finite element method for elliptic problems, North Holland Publ., 1978.
7 A. Chkifa, A. Cohen, R. DeVore and Ch. Schwab, Sparse adaptive Taylor approximation algorithms forparametric and stochastic elliptic PDEs, Modl. Math. Anal. Numr. **47**(1)(2013), 253–280.
8 A. Chkifa, A. Cohen and C. Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs. Found. Comp. Math **14**(2014), 601–633.
9 A. Cohen and R. DeVore, Approximation of high-dimensional parametric PDEs, Acta Numerica **24**(2015), 1–159.
10 A. Cohen, R. DeVore and Ch. Schwab, Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs, Found. Comput. Math. **10**(2010), 615–646 .
11 A. Cohen, R. DeVore and Ch. Schwab, Analytic regularity and polynomial approximation of parametric and stochastic PDEs, Anal. Appl. **9** (2010), 11–47.
12 D. Dũng, Linear collective collocation and Galerkin approximations for parametric and stochastic elliptic PDEs, arXiv:1511.03377, 2015.
13 D. Dũng and M. Griebel, Hyperbolic cross approximation in infinite dimensions, J. Complexity **33**(2016), 33–88.
14 D. Dũng, M.Griebel, N.H. Vu and C. Rieger, ε-dimension in infinite dimensional hyperbolic cross approximation and application to parametric elliptic PDEs, J. Complexity, 46 (2018) 66-89.
15 D. Dũng, V. Temlyakov and T. Ullrich, Hyperbolic Cross Approximation, Series "Advanced Courses in Mathematics, CRM Barcelona", Birkhäuser, Basel, 2018.
16 Ph. Frauenfelder, Ch. Schwab and R.A. Todor, Finite elements for elliptic problems with stochastic coefficients, Comput. Methods Appl. Mech. Eng. **194**(2005), 205–228 .
17 C.J. Gittelson, An adaptive stochastic Galerkin method for random elliptic operators, Math. Comp. **82**(2013), 1515–1541.
18 M. Gunzburger, C. Webster and G. Zang, Stochastic finite element methods for partial diferential equations with random input data, Acta Numerica **23**(2014), 521–650.
19 V.H. Hoang and C. Schwab, N-term Galerkin Wiener chaos approximation rates for elliptic PDEs with lognormal Gaussian random inputs, Math. Models Methods Appl. Sci. **24** (2014) 797–826.
20 H. G. Matthies and A. Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Engrg., **194**(2005), 1295–1331.
21 V. Nistor and Ch. Schwab, High order Galerkin approximations for parametric, second order ellipticpartial differential equations, Math. Models Methods Appl. Sci. **123**(2013), 1729–1760.
22 Ch. Schwab and C. Gittelson, Sparse tensor discretizations high-dimensional parametric and stochastic PDEs, Acta Numerica **120**(2011), 291–467.
23 J. Zech, D. Dũng and Ch. Schwab, Multilevel approximation of parametric and stochastic PDEs, Research Report No. 2018-05, Seminar for Appied Mathematics, ETH.

Сведения об авторах:

Динь Зунг - физика-математика ғылымдарының докторы, проф., Информациялық технологиялар институты, Вьетнам ұлттық университеті, Суан Тхюи 144, Каузяй, Ханой, Вьетнам.

Dinh Dũng - Doctor of Phys.-Math. Sciences, Prof., Information Technology Institute, Vietnam National University, 144 Xuan Thuy, Cau Giay, Hanoi, Vietnam.

Received 29.03.2018