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UNIFORMLY CONVEX SUBSPACES OF MEASURES WITH THE KANTOROVICH NORM

Abstract: In this paper, we consider signed Borel measures on a compact metric space. We study the uniform convexity of the Kantorovich norm on subspaces of the whole space of signed measures. We construct an example of an infinite-dimensional subspace of measures on which the Kantorovich norm is uniformly convex. We also obtain an example of an infinite compact set (X, ρ) such that all uniformly convex subspaces of the space of measures on X are finite-dimensional.

Keywords: Kantorovich norm, uniformly convex space, subspace of measures, borel signed measures.

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1. Introduction

The main object of this paper is the Kantorovich norm, so we start with its definition. Let (X, ρ) be a metric space. Consider the linear space $\mathcal{M}_0(X)$ of all signed Borel measures σ on X such that $\sigma(X) = 0$ and the function $x \to \rho(x, x_0)$ is integrable with respect to the total variation $|\sigma|$ of σ for all $x_0 \in X$.

Definition 1. The Kantorovich norm is the norm $\|\cdot\|_K$ on the space $\mathcal{M}_0(X)$ defined by

$$\|\mu\|_K = \sup\left\{\int_X f d\mu \colon f \in Lip^1(X)\right\}, \quad \mu \in \mathcal{M}_0(X),$$

where

$$Lip^1(X) = \left\{f \colon X \to \mathbb{R}, |f(x) - f(y)| \le \rho(x, y)\ \forall x, y \in X\right\}.$$

In this paper we consider the uniform convexity of the Kantorovich norm on subspaces of the space $\mathcal{M}_0(X)$. First, we show that in general the Kantorovich norm on all of $\mathcal{M}_0(X)$ is not uniformly convex. Next, we prove that in case of measures on an interval the Kantorovich norm is uniformly convex on some infinite-dimensional subspace of the space of measures. Finally, we give an example of an infinite compact set for which the Kantorovich norm is not uniformly convex on any infinite-dimensional subspace of measures.

Let us mention an important result on isometric embeddings (see Corollary 1 on p. 311 in [1]), which we will use below.

Theorem 1. *If $1 \le p \le q \le 2$, then the space $L^q[0,1]$ is isometric to a subspace in $L^p[0,1]$.*

Let us also state two classical results on uniform convexity.

Theorem 2. *If* $1 < p < \infty$ *, then the space* $L^p[0,1]$ *is uniformly convex.*

Theorem 3 (Milman–Pettis)**.** *Every uniformly convex Banach space is reflexive.*

2. MAIN RESULTS

We note that $\mathcal{M}_0(X)$ with the Kantorovich norm is not, in general, a uniformly convex space. This follows obviously from the lemma below. This lemma also gives a necessary condition for the strict convexity of a subspace of measures. Intuitively, this condition means that the subspace should not contain measures with supports that are "far" from each other.

Lemma 1. *Let two measures* $\mu, \nu \in \mathcal{M}_0(X)$ *be given. Suppose that there are two balls* $B_{r_1}(a)$ *and* $B_{r_2}(b)$ *in* X *such that* $supp(\mu) \subset B_{r_1}(a), supp(\nu) \subset B_{r_2}(b)$ *and* $\rho(a,b) > 3(r_1 + r_2)$ *. Then*

$$\|\mu\|_K + \|\nu\|_K = \|\mu + \nu\|_K.$$

P r o o f. Take $f_k \in Lip^1(B_{r_1}(a))$ and $g_k \in Lip^1(B_{r_2}(b))$ such that

$$\left| \int_{B_{r_1}(a)} f_k d\mu - \|\mu\|_K \right| < \frac{1}{k}$$

and

$$\left| \int_{B_{r_2}(b)} g_k d\nu - \|\nu\|_K \right| < \frac{1}{k}.$$

We can assume that $\min_X f_k = 0$, since f_k can be replaced by $f_k - c$ and the integral

$$\int_X f_k d\mu$$

does not change. Then, due to the Lipschitz property, we have $\left|\max_X |f_k| - 0\right| \leq 2r_1$.

Similarly, $\left|\max_X |g_k|\right| \leq 2r_2$. Then for for all $x \in B_{r_1}(a)$ and all $y \in B_{r_2}(b)$ we have

$$\left|f_k(x) - g_k(y)\right| \leq \left|f_k(x)\right| + \left|g_k(y)\right| \leq 2r_1 + 2r_2 < \rho(a,b) - \rho(a,x) - \rho(b,y) \leq \rho(x,y).$$

Next we use the Tietze extension theorem and construct a function $h_k \in Lip^1(X)$ such that for all $x \in B_{r_1}(a)$ we have $h_k(x) = f_k(x)$ and for all $y \in B_{r_2}(b)$ we have $h_k(y) = g_k(y)$.

Hence

$$\left| \int_X h_k d(\mu + \nu) - \|\mu\|_K - \|\nu\|_K \right| < \frac{2}{k}.$$

Therefore, we have

$$\|\mu + \nu\|_K \geq \sup_{h \in Lip^1(X)} \int_X h d(\mu + \nu) \geq \int_X h_k d(\mu + \nu) \geq \|\mu\|_K + \|\nu\|_K - \frac{2}{k}.$$

If we let $k \to \infty$, then we get

$$\|\mu + \nu\|_K \geq \|\mu\|_K + \|\nu\|_K$$

which completes the proof.

We now consider our question for an interval and show that for measures on it there exists an infinite-dimensional uniformly convex subspace. However, first we make a remark about the calculation of the Kantorovich norm in case of measures on an interval.

Remark 1. The Kantorovich norm for measures $\mu \in \mathcal{M}_0[a,b]$ can be calculated as the L^1-norm for their distribution functions.

Indeed, we use the formula (see [1])

$$d_K(P_1, P_2) = \int_{-\infty}^{\infty} \left|\Phi_{P_1}(t) - \Phi_{P_1}(t)\right| dt,$$

which is valid for probability distributions P_1, P_2 and their distribution functions Φ_{P_1}, Φ_{P_2}. We have the Jordan decomposition of $\mu \in \mathcal{M}_0[a,b]$ into positive and negative parts. Using it and, if necessary, normalizing its components, we obtain

$$\|m\|_K = \|m_+ - m_-\|_K = \int_{[a,b]} \big|m_+[a,t] - m_-[a,t]\big| dt = \int_{[a,b]} \big|m[a,t]\big| dt.$$

Theorem 4. *There exists an infinite-dimensional subspace in the space $\mathcal{M}_0[0,1]$ that is uniformly convex in the Kantorovich norm.*

P r o o f. By Theorem 1 there exists a space $Y \subset L^1[0,1]$ isometric to $L^p[0,1]$ if $1 \le p \le 2$. By Theorem 2 the space Y is uniformly convex. It follows from Remark 1 that it suffices to take any linear subspace spanned by vectors from Y regarded as distribution functions for the required space of measures.

So, we have an example of a compact set for which there exists a uniformly convex infinite-dimensional subspace of the space of measures. Of course, not all compacts have this property. For example, for finite compact sets, the space of measures is finite-dimensional and, therefore, cannot contain any infinite-dimensional subspaces. However, there is a stronger counterexample. But before constructing it, we make one more useful remark about the Kantorovich norm.

Remark 2. Let $Y \subset X$. Then $\mathcal{M}_0(Y)$ is isometrically embedded into $\mathcal{M}_0(X)$.

Indeed, by definition, we have

$$\|\mu\|_{K,X} = \sup\left\{ \int_X f d\mu \colon f \in Lip^1(X) \right\}.$$

Consider an arbitrary function $f \in Lip^1(X)$. Since $\mu \in \mathcal{M}_0(Y)$, we have

$$\int_X f d\mu = \int_Y f d\mu.$$

Using the fact that $f\big|_Y \in Lip^1(Y)$ we get

$$\|\mu\|_{K,Y} = \sup\left\{ \int_Y f d\mu \colon f \in Lip^1(Y) \right\} \ge \sup\left\{ \int_Y f\big|_Y d\mu \colon f \in Lip^1(X) \right\} \qquad (1)$$

and

$$\sup\left\{ \int_Y f\big|_Y d\mu \colon f \in Lip^1(X) \right\} = \|\mu\|_{K,X}.$$

In fact, in expression (1), the equality holds. This is true, since by the Tietze theorem any Lipschitz function on Y can be extended to a Lipschitz function on X.

Theorem 5. *There is a compact space (X,ρ) with an infinite-dimensional space $\mathcal{M}_0(X)$ such that every infinite-dimensional closed subspace $Y \subset \mathcal{M}_0(X)$ is not uniformly convex.*

P r o o f. For X we take the following subset of the real line:

$$\left\{ -\frac{1}{n},\ n \in \mathbb{N} \right\} \cup \{0\}.$$

Then any measure $\mu \in \mathcal{M}_0(X)$ has the form

$$\mu = \left(\sum_{n=1}^{\infty} k_n \delta_{-\frac{1}{n}} \right) - \left(\sum_{n=1}^{\infty} k_n \right) \delta_0,$$

where k_n are constant coefficients such that the first moment is finite, i.e.,

$$\sum_{i=1}^{\infty} \frac{k_i}{i} < \infty,$$

and the measure is well-defined by the series

$$\sum_{i=1}^{\infty} k_i < \infty.$$

The Kantorovich norm is calculated using Remarks 1 and 2, because

$$\|\mu\|_K = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)\left|\sum_{i=1}^{n} k_i\right|.$$

Consider the mapping $F\colon \mathcal{M}_0(X) \to l_1$ defined as follows:

$$F(\mu) = \big(a_1, a_2 \dots a_n \dots\big),$$

where

$$a_n = \left(\frac{1}{n} - \frac{1}{n+1}\right)\left(\sum_{i=1}^{n} k_i\right).$$

By definition, the mapping F is an isometric embedding of the metric space $\mathcal{M}_0(X)$ into l_1. Thus, it suffices to show that there are no infinite-dimensional closed uniformly convex subspaces in l_1.

Let $Y \subset l_1$ be a uniformly convex closed subspace. Then, by the Milman–Pettis theorem, Y is reflexive. This means that the closed unit ball B_Y is weakly compact, and hence sequentially weakly compact. Since weak convergence in l^1 implies convergence in norm, our ball B_Y is compact in the norm of l^1, and hence Y is finite-dimensional.

Now let us construct an example of a strictly convex infinite-dimensional subspace "by bear hands". For this purpose, we are going to prove an auxiliary lemma.

Lemma 2. *There are two measures $\mu, \nu \in \mathcal{M}_0[a, b]$ on the interval $[a, b]$ whose linear span is a strictly convex space with the Kantorovich norm.*

P r o o f. Without loss of generality, we can assume that we are solving the problem for the interval $[-3, 3]$ (to obtain the general case, it is enough to shift and scale the interval). Consider the measures μ_1 and μ_2 given by their distribution functions as follows:

$$F_1(x) = \begin{cases} x+3 & \text{if } x \in [-3,-2] \\ -x-1 & \text{if } x \in [-2,0] \\ x-1 & \text{if } x \in [0,2] \\ -x+3 & \text{if } x \in [2,3] \end{cases}$$

$$F_2(x) = \begin{cases} 0 & \text{if } x \in [-3,-2] \\ x+2 & \text{if } x \in [-2,-1] \\ -x & \text{if } x \in [-1,1] \\ x-2 & \text{if } x \in [1,2] \\ 0 & \text{if } x \in [2,3]. \end{cases}$$

We will prove that the span of these two measures is strictly convex. It suffices to check the strict triangle inequality for two non-proportional linear combinations μ_1 and μ_2. In other words, it follows from Remark 1 that for linear combinations $k_1\mu + k_2\nu$ and $l_1\mu + l_2\nu$ it is necessary to verify the following inequality:

$$\int_{[-3,3]} |k_1F_1 + k_2F_2| + |l_1F_1 + l_2F_2| - |(k_1+l_1)F_1 + (k_2+l_2)F_2| dx > 0.$$

This inequality holds if the following is true on a set of nonzero measure:

$$|k_1F_1 + k_2F_2| + |l_1F_1 + l_2F_2| - |(k_1+l_1)F_1 + (k_2+l_2)F_2| dx > 0. \tag{2}$$

The last inequality turns into the equality only if the signs of the expressions $k_1F_1 + k_2F_2$ and $l_1F_1 + l_2F_2$ coincide.

Let us find the zeros of $k_1F_1 + k_2F_2$. We have

$$\left[\begin{array}{lll} \text{for } x \in [-3,-2]\colon & k_1(x+3) = 0 & \iff x = -3; \\ \text{for } x \in [-2,-1]\colon & k_1(-x-1) + k_2(x+2) = 0 & \iff x = -1 - \dfrac{k_2}{k_2 - k_1}; \\ \text{for } x \in [-1,0]\colon & k_1(-x-1) - k_2 x = 0 & \iff x = -\dfrac{k_1}{k_1 + k_2}; \\ \text{for } x \in [0,1]\colon & k_1(x-1) - k_2 x = 0 & \iff x = \dfrac{k_1}{k_1 - k_2}; \\ \text{for } x \in [1,2]\colon & k_1(x-1) + k_2(x-2) = 0 & \iff x = 1 + \dfrac{k_2}{k_1 + k_2}; \\ \text{for } x \in [2,3]\colon & k_1(3-x) = 0 & \iff x = 3. \end{array}\right.$$

It is clear from these expressions that for non-proportional pairs (k_1, k_2) and (l_1, l_2) there is an interval on which the signs of $k_1F_1 + k_2F_2$ and $l_1F_1 + l_2F_2$ are different. So, we obtain that (2) is satisfied on this interval, which proves our claim.

Lemma 3. *Let μ_n be a sequence of measures such that $\mu_n \in \mathcal{M}_0(\mathbb{R})$ and*

$$\mu_n\big|_{(a,b)} = 0$$

for some interval (a,b).
If the sequence of measures μ_n converges to the measure $\mu \in \mathcal{M}_0(\mathbb{R})$ in the Kantorovich norm, then $\mu\big|_{(a,b)} = 0$.

P r o o f. The restrictions of the distribution functions of measures μ_n to the interval (a,b) are equal to constants.The sequence of constants converges to a constant in the L_1-norm. Thus, the lemma follows from Remark 1.

Theorem 6. *There exists a countable family of measures μ_n on the interval $[0,1]$ such that their closed linear span is a strictly convex space with the Kantorovich norm.*

P r o o f. Consider the family of intervals

$$A_k = \big[a_k, b_k\big] = \left[\frac{1}{4^k} - \frac{1}{10 \cdot 4^k}, \frac{1}{4^k} + \frac{1}{10 \cdot 4^k}\right].$$

Using Lemma 2, we construct measures α_k and β_k such that $supp(\alpha_k) \subset A_k$, $supp(\beta_k) \subset A_k$ and the linear span of these measures is a strictly convex space with the Kantorovich norm.

We now construct the desired family of measures μ_n. To this end, we take a bijection s between unordered pairs (n,l) of different indices of measures μ_n and even indices $2k$ of the intervals A_{2k}. Then we set

$$\mu_n = \gamma_n + \sum_{i=1}^{n-1} \alpha_{s(n,i)} + \sum_{i=n+1}^{\infty} \beta_{s(n,i)},$$

where γ_n is defined as

$$\gamma_n = \delta_{a_{2n-1}} - \delta_{b_{2n-1}}.$$

Consider any measure μ lying in the closed linear span of μ_n. We prove that the measure μ has the form

$$\mu = \sum_{i=1}^{\infty} c_i \mu_i, \tag{3}$$

where the series converges to μ in the Kantorovich norm.

Since the measure μ lies in the closed linear span of μ_n, for every ε there is a linear combination $c_1\mu_1 + \cdots + c_j\mu_j$ such that:

$$\|\mu - (c_1\mu_1 + \cdots + c_j\mu_j)\| < \varepsilon$$

Lemma 3 implies the equality

$$\mu\big|_{(b_{k-1}, a_k)} = 0 \tag{4}$$

for each k.

Using (4), we successively apply Lemma 1 to restrict the measure μ to the intervals $[0, b_2]$ and $[a_1, b_1]$, then to the intervals $[0, b_3]$ and $[a_2, b_2]$, ..., $[0, b_{k+1}]$ and $[a_k, b_k]$ etc. Thus, we have:

$$\|\mu\| = \sum_{j=1}^{\infty} \|m_j\|, \tag{5}$$

where m_j is the restriction of μ to $[a_j, b_j]$.

Consider the interval A_{2k+1}. On it, the restrictions of our measures form a one-dimensional space, which enables us to determine the coefficient c_k in representation (3).

Let us prove that for any ν and μ lying in the closed linear span of μ_n and $\nu \neq c\mu$ the inequality $\|\nu\| + \|\mu\| > \|\nu + \mu\|$ is true. Consider a pair of indices $i < j$ for which the measures μ_i and μ_j enter the expansion (see (3)) of the measures ν and μ with non-proportional pairs of coefficients (a, b) and (c, d), respectively. Then from (5) we have

$$\|\nu\| = \|\nu - a\alpha_{s(i,j)} - b\beta_{s(i,j)}\| + \|a\alpha_{s(i,j)} + b\beta_{s(i,j)}\|$$

and

$$\|\mu\| = \|\mu - c\alpha_{s(i,j)} - d\beta_{s(i,j)}\| + \|c\alpha_{s(i,j)} + d\beta_{s(i,j)}\|.$$

Using the strict convexity of the linear span of the measures $\alpha_{s(i,j)}$ and $\beta_{s(i,j)}$ we have

$$\|a\alpha_{s(i,j)} + b\beta_{s(i,j)}\| + \|c\alpha_{s(i,j)} + d\beta_{s(i,j)}\| > \|(a+c)\alpha_{s(i,j)} + (b+d)\beta_{s(i,j)}\|.$$

So, applying the triangle inequality, we obtain what is required.

3. CONCLUSION

In this work, we study the existence of infinite-dimensional uniformly convex subspaces of the space of measures with the Kantorovich norm on a compact set. We show that in case of an interval there are such subspaces. However, an example of a compact set is given for which there are no such subspaces. A restriction necessary for the existence of such subspaces is also established: they must not contain measures with supports that are "far" from each other. Furthermore, a constructive example is given of an infinite-dimensional space of measures that is strictly convex.

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Канторович нормалы бірқалыпты дөңес өлшемдер ішкі кеңістіктері

Аннотация: Мақалада компактты метрикалық кеңістіктегі борелдік өлшемдер қарастырылады. Барлық өлшемдер жиынының жиыншаларында Канторович нормасының бірқалыпты дөңестігі қарастырылады. Канторович нормасы бірқалыпты дөңес болатындай шексіз өлшемді өлшемдердің ішкі кеңістігі құрылды. Барлық бірқалыпты дөңес өлшемдер кеңістігінің ішкі кеңістіктері X-та ақырлы өлшемді болатындай (X, ρ) ақырсыз компактының мысалы алынды.

Түйін сөздер: Канторович нормасы, бірқалыпты дөңес кеңістік, өлшемдер жиыншасы, борелдік өлшемдер.

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Равномерно выпуклые подпространства мер с нормой Канторовича

Аннотация: В работе рассматриваются борелевские меры на компактном метрическом пространстве. Изучается равномерная выпуклость нормы Канторовича на подпространствах всего пространства мер. Построен пример бесконечномерного подпространства мер на котором норма Канторовича равномерно выпукла. Также получен пример бесконечного компакта (X, ρ) такого, что все равномерно выпуклые подпространства пространства мер на X конечномерны.

Ключевые слова: Норма Канторовича, равномерно выпуклое пространство, подпространство мер, борелевские меры.

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