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THE INFORMATIVE POWER OF ALL POSSIBLE LINEAR FUNCTIONALS AND THE MEAN-SQUARE ERROR IN THE DISCRETIZATION OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION IN THE CIRCLE

Abstract: The Dirichlet problem for the Laplace equation in the case of a circle belongs to the classical ones and in various aspects has been the subject of study in various fields of mathematics. Among them are such topics as

- "Boundary properties of analytic functions", in the study of which powerful methods of function theories were created and honed,
- The Banach problem on the existence of a basis for a class of functions consisting of continuous in a closed circle and analytic in,
- Numerical methods, since this problem as a mathematical model describes many real processes.

In this article, we consider the discretization problem of solutions of the Dirichlet problem for the Laplace equation in a circle from finite numerical information obtained from the boundary function as a result of applying all possible linear functionals. The optimal order of discretization error is found and the corresponding optimal operator of discretization is constructed.

The problem of constructing probabilistic measures on functional classes is also considered. Probabilistic measures on the Korobov $E^r(0, 2\pi)$ and Nikolsky $H_2^r(0, 2\pi)$ classes are introduced. Two-sided estimates of the mean-square error of discretization the solution of the problem by operator $(T_N f)(\alpha, \theta)$ are established.

Keywords: informative power of a given class of functionals; discretization of solutions of a differential equation; mean-square error in relation to a probability measure

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1. INTRODUCTION

The statement of the problem, which is the subject of this work, is preceded by an extensive citation from the article [1] of P. Laks:

"Everyone knows about the incredible progress made over the past 50 years in the speed of computers and the amount of information they store, as well as improvements in graphics and software. As a result, tasks that were previously on the edge of the computer's capabilities can now be solved much faster and cheaper, and we can approach tasks of daunting complexity. But what many people don't realize is that much of this progress is due not only to improvements in hardware and software, but equally to new mathematical ideas about how to solve emerging computational problems.

Here are some amazing examples.

Multi-grid method. After discretizing elliptic systems of partial differential equations - the standard example is the Dirichlet problem for the Laplace equation-the problem of numerical solution of the resulting system of linear algebraic equations arises. An effective iterative method for performing this task, called the multigrid method, was proposed in the 1960s by R.P. Fedorenko [2] and analyzed by N.S. Bakhvalov [3]; it was further developed and applied by Aki Brandt [4]. Schematically, the idea is to obtain information about the behavior of the solution over large distances by computing on a coarse grid.

The purpose of this paper is to elucidate the approximative capabilities of computational aggregates constructed using arbitrary algorithms applied to numerical information of a given volume N obtained from boundary functions by means of N linear functionals in the approximation problem (in the L^q metric) of the solution of the Dirichlet problem for the Laplace equation.

We present the statement of the general problem (in edition [5, 6]) in relation to the concretization considered in this paper.

Let $u(\alpha,\theta) \equiv u(\alpha,\theta;f)$ be a solution of the Laplace equation in polar coordinates (see [7, page 236])

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{1}{\alpha}\frac{\partial u}{\partial \alpha} + \frac{1}{\alpha^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \tag{1}$$

satisfying the boundary condition

$$u(\alpha,\theta)|_{\alpha=R} = f(\theta), \tag{2}$$

where - f is some 2π - periodic function that ensures the correctness of the problem (1)-(2), moreover, the belonging of the solution $u(\alpha,\theta;f)$ to a given normalized space Y.

For each positive integer N by $L_N \equiv \{(l^{(N)};\varphi_N)\}$ we denote the set of all pairs $(l^{(N)};\varphi_N)$, where $l^{(N)} = (l_1,\ldots,l_N)$ is an ordered set of linear functionals $l_j(\cdot)$ $(j = 1,\ldots,N)$, defined on the linear envelope of a set F, and the function $\varphi_N \equiv \varphi_N(\tau_1,\ldots,\tau_N;\alpha,\theta)$ acts from $C^N \times [0,R] \times [0,2\pi]$ in C, where C, as usual, the field of complex numbers.

Also assume that for arbitrary fixed $\tau_1,\ldots,\tau_N$ and α function φ_N, considered as a function of θ, belongs to a normed space Y.

For $\left(l^{(N)};\varphi_N\right) \in D_N \subset L_N$ assume

$$\delta_N((l^{(N)};\varphi_N);\, F)_Y = \sup_{f\in F}\sup_{0\le\alpha\le R} \|u(\alpha,\cdot,f) - \varphi_N(l_1(f),...,l_N(f),\alpha,\cdot)\|_Y$$

and

$$\delta_N(D_N;F)_Y = \inf_{\left(l^{(N)};\varphi_N\right)\in D_N} \delta_N((l^{(N)};\varphi_N);\, F)_Y. \tag{3}$$

In the case of $A_N \subset L_N,\ D_N = A_N \times \{\varphi_N\}$ value (3) is called the informative power of a given class of functionals A_N, and in the case of $A_N = L_N$ - the informative power of all possible linear functionals.

The task is to obtain the upper and lower estimates for the value (3) and to specify the pair $(l^{(N)};\varphi_N)$ that implements the upper estimate.

From a position of evaluating the results of this article it is important to note (see [5, 6, 8]) that the linear diameter, diameter Fourier transform (orthopaedic), the partial sums of the Fourier series over all orthonormal systems (including systems consisting of bursts) and decomposition on the bases, linear methods of summation of Fourier series when the appropriate choice $D_N \subset L_N$ is also contained in the definition (3), and for every choice D_N holds the inequality

$$\delta_N(D_N;F)_Y \ge \delta_N(L_N;F)_Y.$$

Also note that the problem of reconstruction in the above statement is devoted to the work [5, 6, 8–20].

In this paper, as a class F considered the Sobolev class $W_q^r(0,2\pi)$ (r - is positive integer), defined as follows (see [21, 22]):

$$f(\theta) \in W_q^r(0,2\pi) \overset{def}{\Leftrightarrow} \left\| \left\{ \hat{f}(m)\cdot \overline{m}^r \right\}_{m=-\infty}^{+\infty} \right\|_{l^q} \le 1,$$

and the Besov class $B^r_{q,æ}(0,2\pi)$ (see [21]):

$$f(\theta) \in B^r_{q,æ}(0,2\pi) \overset{def}{\Leftrightarrow} \left\| \left\{ 2^{\tau r} \left\| \sum_{m\in\rho(\tau)} \hat{f}(m) \cdot e^{im\cdot} \right\|_{L_q} \right\}_{\tau=1}^{\infty} \right\|_{l^æ} \leq 1,$$

where $(1 \leq q, æ \leq \infty, r > 0)$, $\overline{m} = \max\{1;\ |m|\}, \rho(\tau) = \left\{m \in Z:\ 2^{\tau-1} \leq \overline{m} < 2^{\tau}\right\}$, $\|f\|_{L_q}$ - the Lebesgue norm with the q degree of summability of the 2π- periodic function f,

$$\hat{f}(m) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cdot e^{-im\theta} d\theta$$

-the trigonometric Fourier coefficients of the function f, and $\|\{a_\tau\}_{\tau=1}^{\infty}\|_{l^æ}$ is the norm of the numerical sequence $a = \{a_\tau\}_{\tau=1}^{\infty}$, defined as

$$\|\{a_\tau\}_{\tau=1}^{\infty}\|_{l^æ} = \begin{cases} \left(\sum_{\tau=1}^{\infty} |a_\tau|^æ\right)^{\frac{1}{æ}} & \text{if } 1 \leq æ < \infty, \\ \sup\limits_{\tau} |a_\tau| & \text{if } æ = \infty. \end{cases}$$

By $c(\dots)$ we will denote some positive values that are different, generally speaking, in different formulas and depend only on the parameters specified in parentheses. With a positive A and any B record $B = O_{\alpha,\beta,\dots}(A), B \underset{\alpha,\beta,\dots}{\ll} A$ will mean $|B| \leq c(\alpha,\beta,\dots)A$. With a positive A and B record $A \underset{\alpha,\beta,\dots}{\asymp} B$ means $A \underset{\alpha,\beta,\dots}{\ll} B \underset{\alpha,\beta,\dots}{\ll} A$.

In the following two paragraphs, the problems of discretization of solutions of equation (1)-(2) are investigated in the sup-norm and in the root-mean-square, respectively.

2. DISCRETIZATION IN THE SUP-NORM

The following theorem holds.

Theorem 1. *Let the numbers* $2 \leq q \leq \nu \leq +\infty$ *be given.*

a) Let r *- be a positive integer. Then there is a two-sided estimate* $(N = 1, 2, \dots)$

$$\delta_N(L_N, W^r_q(0,2\pi))_{L^\nu[0,2\pi]} \asymp N^{-\left(r-\left(\frac{1}{q}-\frac{1}{\nu}\right)\right)}.$$

в) Let $rq > 1$ *and* $1 \leq æ \leq \infty$. *Then there is a two-sided estimate* $(N = 1, 2, \dots)$

$$\delta_N(L_N, B^r_{q,æ}(0,2\pi))_{L^\nu[0,2\pi]} \asymp N^{-\left(r-\left(\frac{1}{q}-\frac{1}{\nu}\right)\right)}.$$

Wherein in each of the cases a) and в) the upper estimate is implemented by the operator $(N = 2^n, n = 1, 2, \dots)$

$$\varphi_N(l_1(f), \dots, l_N(f); \alpha, \theta) = V_{2^n}(\alpha, \theta; f) = \sum_{k=0}^{n} Q_k(\alpha, \theta; f),$$

where $V_{2^k}(\alpha,\theta;f)$ *- the average Vallée Poussin function* $u(\alpha,\theta;f)$ *of order* 2^k *by variable* θ, *and* $Q_0 = V_{2^0}, Q_k = V_{2^k} - V_{2^{k-1}}$ *for all* $k \geq 1$ *[21, p. 295-300].*

3. DISCRETIZATION IN THE MEAN SQUARE

The main way to assess the quality of the reconstruction is the "worst case", i.e. the case when the maximum class error is calculated. At the same time, the comparison of operators for reconstruction functions from a class by the maximum (in terms of the class) error (deviation) can turn out to be rough: two operators can have the same maximum deviations, while for the first operator it is achieved on functions that are in a certain sense “few” in the class, and for the second-on “most” functions of the class (quantitative characterization of the sizes of subsets of functions is made on the basis of the Lebesgue measure). Although the first sampling method is obviously preferable, the two methods are indistinguishable when evaluating the quality of the approximation by the value of the maximum deviation.

Thus, the statement of this problem consists in estimates from below and from above of the value (see also (3))

$$\int\limits_{F} \sup_{0\leq\alpha\leq R} \left\| u\left(\alpha,\cdot,f\right) - \varphi_N\left(l_1\left(f\right),...,l_N\left(f\right),\alpha,\cdot\right)\right\|_Y^2 d\mu_F(f),$$

$\mu_F(f)$ is a probability measure defined on F.

Let $\{\Gamma_k\}$ be a sequence of pairwise disjoint finite sets $\Gamma_k \subset Z$, the union of which is all Z. By d_k we denote the number of points in the Γ_k.

Let $\nu_{-1} = 0$ and $\nu_k = d_0 + \cdots + d_k, a_{j(m)}$ are a fixed ordering of Γ_k. Then each set

$$Y = \{y_m\}_{m\in Z} \tag{4}$$

complex numbers, taking into account the equality, $y_m = a_j(m) + ib_j(m)$, we will consider it represented as a sequence $Y = (a_1, b_1, \ldots, a_{\nu_k}, b_{\nu_k}, \ldots)$.

Next, let for each k on a $2d_k$ – dimensional Euclidean space R^{2d_k} - be given a non-negative continuous function ψ_k such that $\psi_k(0) = 0$.

We define the classes $H(\Gamma_k, \psi_k)$ as the collection of all sets (4) such that for each $k(k = 0, 1, 2, \ldots)$ the inequality is satisfied

$$\psi_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) \leq 1,$$

i.e.

$$H(\Gamma_k, \psi_k) = \{(a_1, b_1, \ldots, a_{\nu_k}, b_{\nu_k}, \ldots) : \psi_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) \leq 1\}.$$

Suppose

$$H \stackrel{def}{=} \bigcap_{k=1}^{\infty} H(\Gamma_k, \psi_k).$$

Let

$$D_k = \{(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) \in R^{2d_k} : \psi_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) \leq 1\}.$$

Then

$$H = D_0 \times D_1 \times \cdots \times D_k \times \cdots$$

Cylindrical sets $T_k(E_k)$ are defined as follows:

$$T_k(E_k) = \{(a_1, b_1, \ldots, a_{\nu_k}, b_{\nu_k}, \ldots) \in H : (a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) \in E_k\},$$

where $E_k \subset D_k$ $(E_k \in F(D_k))$.

Let $F(H)$ – be the smallest σ-algebra containing all cylindrical sets.

Theorem A (see [23]). *Each of the possible probability measures μ on the measurable space $(H, \mathcal{F}(H))$ is uniquely determined by setting the sequence of measures μ_k on $(D_k, \mathcal{F}(D_k))$ such that for all $k(k = 0, 1, 2, \ldots)$ an $E_k \in \mathcal{F}(D_k)$ the next equality holds*

$$\mu(T_k(E_k)) = \mu_k(E_k).$$

Following this theorem, we introduce a probability measure on the Korobov $E^r(0, 2\pi)$ and Nikolsky $H_2^r(0, 2\pi)$ classes.

By definition of the class $E^r(0, 2\pi)(r > 1)$

$$f(x) \in E^r(0, 2\pi) \Leftrightarrow \left|\hat{f}\left(m\right)\right| \leq \frac{1}{\bar{m}^r} \equiv \rho m(m \in Z),$$

where $f(x)$ - is a 2π-periodic function.

Let for each $m \in Z$ the function

$$\lambda_m(\tau) : [0, \rho_m] \to [0, 1]$$

is continuous, non-decreasing on $[0, \rho_m]$ and satisfies the conditions $\lambda_m(0) = 0$ and $\lambda_m(\rho_m) = 1$.

For any α by $K(\alpha)$ we denote a closed circle of the complex plane with the center at zero and with radius α:

$$K(\alpha) = \{z = \tau e^{i\varphi} : 0 \leq \tau \leq \alpha, 0 \leq \varphi \leq 2\pi\} = \{z \in C : z \leq \alpha\}.$$

Then

$$f(x) \in E^r(0, 2\pi) \Leftrightarrow f(x) = \sum_{m \in Z} \hat{f}\,(m)\, e^{im\theta},\ \hat{f}\,(m) \in K\,(\rho_m)\,.$$

Hence, the mapping

$$E^r \ni f \to \left(\hat{f}\,(0)\,, \hat{f}\,(1)\,, \hat{f}\,(-1)\,, ...\right) \in K\,(\rho_0) \times K\,(\rho_1) \times K\,(\rho_{-1}) \times \cdots$$

establishes a one-to-one correspondence

$$E^r(0, 2\pi) \leftrightarrow K\,(\rho_0) \times K\,(\rho_1) \times K\,(\rho_{-1}) \times \cdots$$

Hence, by virtue of the theorem on setting a measure on the Cartesian product of a countable number of spaces with a measure (see [24, pp. 152-156]), to enter a measure on $E^r(0, 2\pi)$, it is sufficient to enter probabilistic measures μ_m in each $K\,(\rho_m)\,(m \in Z)$. As μ_m, we take a plane Lebesgue measure in a circle $K\,(\rho_m)$ such that if

$$0 \leq \rho_m^{(1)} < \rho_m^{(2)} \leq \rho_m \text{ and } 0 \leq \varphi 1 < \varphi 2 \leq 2\pi,$$

then

$$\mu_m(\tau e^{i\varphi} : \rho_m^{(1)} \leq \tau \leq \rho_m^{(2)}, \varphi 1 < \varphi \leq \varphi 2) = \frac{\varphi_2 - \varphi_1}{2\pi}\left(\lambda_m\left(\rho_m^{(2)}\right) - \lambda_m\left(\rho_m^{(1)}\right)\right). \tag{5}$$

In particular,

$$\mu_m\,(K\,(\rho_m)) = \frac{2\pi}{2\pi}(\lambda_m(\rho_m) - \lambda_m(0)) = 1.$$

Measure μ_m in $K\,(\rho_m)$ with condition (5) is denoted by μ_m^λ.

Then the desired measure in the class $E^r(0, 2\pi)$ itself is defined as follows. Let be given a positive integer n and integers $m^{(1)}, \ldots, m^{(n)}$. For planar μ_m^λ-measurable sets

$$E^{(1)} \subset K\,(\rho_{m^{(1)}})\,, E^{(2)} \subset K\,(\rho_{m^{(2)}})\,, \ldots, E^{(n)} \subset K\,(\rho_{m^{(n)}})$$

consider cylindrical sets

$$T(E^{(3)}, \ldots, E^{(n)}) = \{f \in E^r : \hat{f}\left(m^{(1)}\right) \in E^{(1)}, \ldots, \hat{f}\left(m^{(n)}\right) \in E^{(n)}\} \subset E^r.$$

Then, according to Theorem A, the equalities

$$\mu^\lambda(T(E^{(1)}, \ldots, E^{(n)})) = \prod_{j=1}^{n} \mu_{m^{(j)}}^\lambda(E^{(j)}). \tag{6}$$

the probability measure μ^λ is uniquely determined on the smallest σ - algebra of subsets E^r containing all cylindrical sets $T(E^{(1)}, \ldots, E^{(n)})$.

Next, we give the definition of the probability measure given on $H_2^r(0, 2\pi)$.

Let

$$\rho(\tau) = \left\{m \in Z :\, 2^{\tau-1} \leq \overline{m} < 2^\tau\right\} \equiv \left\{2^{\tau-1}, -\,2^{\tau-1}, 2^{\tau-1} + 1, -\,2^{\tau-1} - 1, ..., 2^\tau - 1,\, -2^\tau + 1\right\}$$

for $\tau = 2, 3, \ldots$ and $\rho(1) = \{0, 1, -1\}$ subsets of Z ("binary bundles") and let $n_\tau = |\rho(\tau)|$. Then by the equivalent definition of class $H_2^r(0, 2\pi)$ (see e.g. [22])

$$f \in H_2^r(0, 2\pi) \;\Leftrightarrow\; \sum_{m \in \rho(\tau)} \left|\hat{f}(m)\right|^2 \leq 2^{-2\tau r}.$$

We introduce the following notation for:

$$f \in H_2^r(0, 2\pi):$$

$$z^{(\tau)}(f) \equiv (\hat{f}(2^{\tau-1}),\ \hat{f}(-2^{\tau-1}), \hat{f}(2^{\tau-1} + 1), \hat{f}(-2^{\tau-1} - 1), ..., \hat{f}(2^\tau - 1),\ \hat{f}(-2^\tau + 1))$$

for $\tau = 2, 3, \ldots$ and $z^{(1)}(f) \equiv (\hat{f}(0),\ \hat{f}(1), \hat{f}(-1))$.

If a ball in C^{n_τ} (or in R^{2n_τ}) of radius $2^{-\tau r}$ centered at the origin is denoted by D_τ , i.e.

$$D_\tau=\left\{z^{(\tau)}=\left(z_1^{(\tau)},z_2^{(\tau)},...,z_{n_\tau}^{(\tau)}\right)\in C^{n_\tau}:\left|z_1^{(\tau)}\right|^2+\left|z_2^{(\tau)}\right|^2+\dots+\left|z_{n_\tau}^{(\tau)}\right|^2\le 2^{-2\tau r}\right\},$$

then the above definition of class $H_2^r(0,2\pi)$ is rewritten as

$$f\in H_2^r(0,2\pi)\Leftrightarrow z^{(\tau)}(f)\in D_\tau\quad(\forall\tau=1,\ 2,...).$$

Hence, taking into account the introduced notation, and the Riesz-Fischer theorem, we can establish the following one-to-one correspondence:

$$H_2^r(0,2\pi)\ni f(x)\leftrightarrow$$

$$\leftrightarrow\overset{\wedge}{(}\underbrace{\hat f(0),\ \hat f(1),\ \hat f(-1)}_{z^{(1)}(f)},\underbrace{\hat f(2),\hat f(-2),\hat f(3),\hat f(-3)}_{z^{(2)}(f)},...,\underbrace{\hat f(2^{\tau-1}),\hat f(-2^{\tau-1}),...,\hat f(2^\tau-1),\hat f(-2^\tau+1)}_{z^{(\tau)}(f)},...)\equiv$$

$$\equiv\left(z^{(1)}(f),\,z^{(2)}(f),\,...,z^{(\tau)}(f),...\right)\in D_1\times D_2\times\cdots\times D_\tau\times\cdots.$$

Let for each $\tau=1,2,...$ measure μ_τ be an absolutely continuous probability measure on $D\tau$:

$$\mu_\tau(E_\tau)=\int_{E_\tau}p_\tau\left(z_1^{(\tau)},z_2^{(\tau)}...,z_{n_\tau}^{(\tau)}\right)dx_1^{(\tau)}dy_1^{(\tau)},...,dx_{n_\tau}^{(\tau)}dy_{n_\tau}^{(\tau)},$$

where $E_\tau\subset D_\tau$ -is a measurable set,

$$z_k^{(\tau)}=x_k^{(\tau)}+iy_k^{(\tau)}\ (k=1,...,n_\tau),$$

and the density

$$p_\tau(z^{(\tau)})=p_\tau(x_1^{(\tau)},y_1^{(\tau)},...,x_{n_\tau}^{(\tau)},y_{n_\tau}^{(\tau)})=p_\tau((x_1^{(\tau)})^2+(y_1^{(\tau)})^2+...+(x_{n_\tau}^{(\tau)})^2+(y_{n_\tau}^{(\tau)})^2)$$

- radially depends on $\left(x_1^{(\tau)},y_1^{(\tau)},...,x_{n_\tau}^{(\tau)},y_{n_\tau}^{(\tau)}\right)\in D_\tau$. Let be given positive integers k and numbers $\tau_1,...,\tau_k(\tau_i\ne\tau_j\text{ at } i\ne j)$. For measurable sets $E_{\tau_k}\subset D_{\tau_k}$ consider cylindrical sets

$$T(E_{\tau_1},...,\ E_{\tau_k})=\left\{f\in H_2^r(0,2\pi):z^{(\tau_1)}(f)\in E_{\tau_1},...,z^{(\tau_k)}(f)\in E_{\tau_k}\right\}\tag{7}$$

Suppose

$$\mu(T(E_{\tau_1},...,E_{\tau_k}))\overset{def}{=}\prod_{i=1}^k\int_{E_{\tau_i}}p_{\tau_i}(z^{(\tau_i)})\,dx^{(\tau_i)}dy^{(\tau_i)}.\tag{8}$$

The set of cylindrical sets of the form (7) forms a semiring.

Then, by Theorem A on the continuation of the measure, we can assume that in the smallest σ- algebra $\mathcal{F}(H_2^r)$, which contains all cylindrical sets of the form (7), the measure μ is given.

In the following theorems, we obtain two-sided estimates for the mean-square discretization error of the solution of problem (1)-(2) regardingto the introduced measures. In both theorems

$$(T_Nf)(\alpha,\theta)=\sum_{i=1}^N f\left(2\pi\frac{i}{N}\right)K_N\left(\alpha,\theta-2\pi\frac{i}{N}\right),$$

where ($[\dots]$ – is the integer part)

$$K_N(\alpha,t)=\frac{1}{N}\left(1+2\cdot\sum_{n=1}^{[N/2]-1}\left(\frac{\alpha}{R}\right)^n\cdot\cos nt\right).$$

Theorem 2. *Let $u(\alpha,\theta;f)$ be the solution of problem (1)-(2), $r>1$ and μ^λ be the probability measure (6), defined on E^r. Then*

$$\int\limits_{E^r(0,2\pi)}\sup_\alpha\|u(\alpha,\cdot;f)-(T_Nf)(\alpha,\cdot)\|_{L^2(0,2\pi)}^2d\mu^\lambda(f)\asymp\sum_{m\in Z\setminus\left(-\left[\frac{N}{2}\right],\left[\frac{N}{2}\right]\right)}\int_0^{\rho_m}\tau^2\lambda_m'(\tau)\,d\tau.$$

Corollary. Let

$$\lambda_m(\tau) = \begin{cases} \frac{\tau(1-\rho_m^2)}{\rho_m^2}, & \text{if } \tau \in \left[0, \rho_m^2\right] \\ \frac{\rho_m\left(\tau-\rho_m^2\right)}{1-\rho_m} + 1 - \rho_m^2, & \text{if } \tau \in \left[\rho_m^2, \rho_m\right]. \end{cases}$$

Then

$$\int\limits_{E^r(0,2\pi)} \sup_{\alpha} \| u(\alpha, \cdot; f) - (T_N f)(\alpha, \cdot) \|^2_{L^2(0,2\pi)} d\mu^\lambda(f) \asymp C(r) \frac{1}{N^{2\left(2r-\frac{1}{2}\right)}}.$$

Theorem 3. *Let* $u(\alpha, \theta; f)$ *- be the solution of problem* (1)-(2), $r > \frac{1}{2}$ *and* μ *- be the probability measure* (8). *Then there is a two-sided estimate* ($N = 2^n$, $n = 1, 2, ...$)

$$\int\limits_{H_2^r(0,2\pi)} \sup_{\alpha} \|u(\alpha, \cdot; f) - (T_N f)(\alpha, \cdot)\|^2_{L^2(0,2\pi)} d\mu(f) \asymp$$

$$\asymp \sum_{\tau=n}^{\infty} \int\limits_{D_\tau} \left(x_1^2 + y_1^2 + ... + x_{n_\tau}^2 + y_{n_\tau}^2\right) p_\tau(x, y) dx dy.$$

4. CONCLUSION

The problem of discretization of solutions by means of arbitrary linear functionals is studied. It is shown that the orders of discretization error are unimprovable.

Regarding probabilistic measures on the Korobov E^r and Nikolsky H_2^r functional classes, two-sided estimates of the mean-square error of discretization the solution of the problem by operator $(T_N f)(\alpha, \theta)$ are established.

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Барлық сызықтық функционалдардың ақпараттық қуаттылығы және дөңгелек жағдайындағы Лаплас теңдеуі үшін Дирихле есебінің шешімін жуықтап калыптастырудағы орташа квадраттық қателік

Аннотация: Дөңгелек жағдайындағы Лаплас теңдеуі үшін Дирихле есебі классикалық есепке жатады және түрлі аспектілерде математиканың әртүрлі салаларында зерттеу нысанын құрды. Олардың ішінде келесі тақырыптарды атап өтуге болады:

- «Аналитикалық функциялардың шекаралық қасиеттері», оларды зерттеу негізінде функциялар теориясының мықты әдістері пайда болды және шыңдалды.

- Дөңгелек ішінде аналитикалық, тұйық дөңгелекте үзіліссіз болатын функциялар класы үшін базистің бар болуы туралы Банах есебі.

- Сандық әдістер, өйткені бұл есеп математикалық модель ретінде көп процесстерді бейнелейді. Осы жұмыста барлық мүмкін сызықты функционалдарды қолдану нәтижесінде шекаралық функциядан алынған ақырлы сандық ақпарат арқылы дөңгелектегі Лаплас теңдеуі үшін Дирихле есебінің шешімін жуықтап қалыптастыру есебі қарастырылған. Жуықтап қалыптастыру қателіктерінің дәл реті табылған және жуықтап қалыптастырудың сәйкес тиімді операторы құрылған. Сонымен қатар функционалдық кластарда ықтималдық өлшемдерін енгізу есебі қарастырылған. Коробов $E^r(0,2\pi)$ және Никольский $H_2^r(0,2\pi)$ класстарына ықтималдық өлшемдері енгізілген. $(T_N f)(\alpha,\theta)$ операторы арқылы шешімді жуықтап қалыптастыру есебінің орташа квадраттық қателігінің екі жақты бағасы алынған.

Түйінді сөздер: берілген функционалды кластың ақпараттық қуаты, дифференциалдық теңдеудің шешімдерін жуықтап қалыптастыру, ықтималдық өлшеміне қатысты орташа квадраттық қателік.

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Информативная мощность всевозможных линейных функционалов и среднеквадратическая погрешность при дискретизации решений задачи Дирихле для уравнения Лапласа в круге

Аннотация: Задача Дирихле для уравнения Лапласа в случае круга относится к классическим и в различных аспектах составляла предмет изучения в разных областях математики. Среди них такие темы, как

- «Граничные свойства аналитических функций», при изучении которых создавались и оттачивались мощные методы теорий функций,

- Проблема Банаха о существовании базиса для класса функций, состоящего из непрерывных в замкнутом круге и аналитических внутри,

- Численные методы, поскольку данная задача как математическая модель описывает многие реальные процессы.

В данной работе рассмотрена задача дискретизации решений задачи Дирихле для уравнения Лапласа в круге по конечной числовой информации, полученной от граничной функции в результате применения всех возможных линейных функционалов. Найден оптимальный порядок погрешности дискретизации и построен соответствующий оптимальный оператор дискретизации.

Также рассмотрена задача построения вероятностных мер на функциональных классах. Введены вероятностные меры на классах Коробова $E^r(0,2\pi)$ и Никольского $H_2^r(0,2\pi)$. Установлены двусторонние оценки среднеквадратической погрешности дискретизации решения задачи посредством оператора $(T_N f)(\alpha,\theta)$.
Ключевые слова: информативная мощность данного класса функционалов, дискретизация решений дифференциального уравнения, среднеквадратическая погрешность относительно вероятностной меры.

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