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On some properties of quasivarieties generated by specific finite modular lattices [1]

Abstract: A finite algebra A with discrete topology generates a topological quasivariety consisting of all topologically closed subalgebras of non-zero direct powers of A endowed with the product topology. This topological quasivariety is standard if every Boolean topological algebra with the algebraic reduct in Q(A) is profinite. In the article it is constructed the specific finite modular lattice T that does not satisfy one of Tumanov's conditions but quasivariety Q(T) generated by this lattice is not finitely based. We investigate the topological quasivariety generated by the lattice T and prove that it is not standard. And we also would like to note that there is an infinite number of lattices similar to the lattice T.

Keywords: lattice, finite lattice, modular lattice, Tumanov's conditions, quasivariety, topological quasivariety, standard topological quasivariety.

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Introduction. The present work considers quasivarieties generated by specific finite modular lattices and investigates their property "to be finitely based" and "to be standard".

According to R. McKenzie [1], any finite lattice has a finite basis of identities. The similar result for quasi-identities is not true, that was established by V.P. Belkin [2]. In 1979 he proved that there is a finite lattice that has no finite basis of quasi-identities. In particular, the smallest lattice that does not have a finite basis of quasi-identities is the ten-element modular lattice M_{3-3}. In this regard, the following question naturally arises. Which finite lattices have finite bases of quasi-identities? This problem was suggested by V.A. Gorbunov and D.M. Smirnov [3] in 1979. V.I. Tumanov [4] in 1984 found sufficient condition consisting of two parts under which the locally finite quasivariety of lattices has no finite (independent) basis for quasi-identities. Also he conjectured that a finite (modular) lattice has a finite basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W. Dziobiak [5] found a finite lattice that generates finitely axiomatizable proper quasivariety. Tumanov's problem is still unsolved for modular lattices.

The paper [6] introduces the concept of a finite standard structure, investigates its basic properties and provides many examples of standard and non-standard structures. The standardness of algebras was further studied by D.M. Clark, B.A. Davey, R.S. Freese and M.G. Jackson in [7], who established a general condition guaranteeing the standardness of a set of finite algebras. Theorem 2.13 from [8] extendes this result. The problem "Which finite lattices generate a standard topological prevariety?" was suggested by D.M. Clark, B.A. Davey, M.G. Jackson and J.G. Pitkethly in the same paper [8]. The paper [9] investigated the questions of the standardness of

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quasivarieties and found sufficient conditions under which a quasivariety contains a continuum of non-standard subquasivarieties without an independent basis of quasi-identities and a continuum of non-standard subquasivarieties with the so-called finitely split basis of quasi-identities.

In this paper we construct a finite modular lattice that does not satisfy one of Tumanov's conditions [4] but the quasivariety generated by this lattice is not finitely based (has no finite basis of quasi-identities). We investigate the topological quasivariety generated by the constructed lattice and prove that it is not standard. And we would like to note that there is an infinite number of lattices similar to this lattice.

Materials and research methods. We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of general algebra introduced below and used throughout this paper, we refer to [10] and [11].

A *quasivariety* is a class of algebras of the same type that is closed with respect to subalgebras, direct products (including the direct product of an empty family), and ultraproducts. Equivalently, a *quasivariety* is the same thing as a class of lattices axiomatized by a set of quasi-identities. A *quasi-identity* means a universal Horn sentence with the non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \cdots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \rightarrow p(\bar{x}) \approx q(\bar{x})],$$

where p, q, $p_1, q_1, \ldots, p_n, q_n$ are lattice's terms. A *variety* is a quasivariety which is closed under homomorphisms. According to Birkhoff theorem [12], a variety is a class of similar algebras axiomatized by a set of identities, where by an identity we mean a sentence of the form $(\forall \bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$.

The smallest quasivariety containing a class $\mathbf{K}$ is denoted by $\mathbf{Q}(\mathbf{K})$. If $\mathbf{K}$ is a finite family of finite algebras then $\mathbf{Q}(\mathbf{K})$ is called finitely generated. If $\mathbf{K} = \{A\}$ we write $\mathbf{Q}(A)$.

Let $\mathbf{K}$ be a quasivariety. A congruence α on algebra A is called a $\mathbf{K}$*-congruence* or *relative congruence* provided $A/\alpha \in \mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}(A)$ of all $\mathbf{K}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$ which is called a *relative congruence lattice.*

The least $\mathbf{K}$-congruence $\theta_{\mathbf{K}}(a, b)$ on algebra $A \in \mathbf{K}$ containing pair $(a, b) \in A \times A$ is called a *principal* $\mathbf{K}$*-congruence* or a *relative principal congruence.* In case when $\mathbf{K}$ is a variety, relative congruence $\theta_{\mathbf{K}}(a, b)$ is usual principal congruence that we denote by $\theta(a, b)$.

An algebra A belonging to a quasivariety $\mathbf{K}$ is *(finitely) subdirectly irreducible relative to* $\mathbf{K}$, or *(finitely) subdirectly* $\mathbf{K}$*-irreducible*, if intersection of any (finite) number of nontrivial $\mathbf{K}$-congruences is again nontrivial; in other words, the trivial congruence 0_A is a (meet-irreducible) completely meet-irreducible element of $\mathrm{Con}_{\mathbf{K}}(A)$.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice L. A pair $(a, b) \in L \times L$ is called *dividing* (*semi-dividing*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-dividing pair (a, b) of a lattice M we define a lattice

$$M_{a-b} = \langle \{(x, 0), (y, 1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge \rangle \leq_s M \times \mathbf{2},$$

where $\mathbf{2} = \langle \{0, 1\}; \vee, \wedge \rangle$ is a two element lattice.

Theorem 1 (Tumanov's theorem [4])**.** *Let* $\mathbf{M}$*,* $\mathbf{N}$ *(*$\mathbf{N} \subset \mathbf{M}$*) be locally finite quasivarieties of lattices satisfying the following conditions:*

a) in any finitely subdirectly $\mathbf{M}$*-irreducible lattice* $M \in \mathbf{M} \backslash \mathbf{N}$ *there is a semi-dividing pair* (a, b) *such that* $M_{a-b} \in \mathbf{N}$*;*

b) there exists a finite simple lattice $P \in \mathbf{N}$ *which is not a proper homomorphic image of any subdirectly* $\mathbf{N}$*-irreducible lattice.*

Then the quasivariety $\mathbf{N}$ *has no coverings in the lattice of subquasivarieties of* $\mathbf{M}$*. In particular,* $\mathbf{N}$ *has no finite basis of quasi-identities provided* $\mathbf{M}$ *is finitely axiomatizable.*

A finite algebra A with discrete topology τ generates a topological quasivariety $\mathbf{Q}_\tau(A)$ consisting of all topologically closed subalgebras of non-zero direct powers of A endowed with the product topology. Profinite algebras are exactly those that are isomorphic to inverse limits of finite algebras. Such algebras are naturally equipped with Boolean topologies. A topology

τ is Boolean if it is compact, Hausdorff, and totally disconnected. A topological quasivariety $\mathbf{Q}_\tau(A)$ is standard if every Boolean topological algebra with the algebraic reduct in $\mathbf{Q}(A)$ is profinite. In this case, we say that algebra A generates a standard topological quasivariety. For more information on the topological quasivarieties we refer to [7] and [8].

Results and discussion. Let T be a modular lattice displayed in Figure 1. And let $\mathbf{N} = \mathbf{Q}(T)$ and $\mathbf{M} = \mathbf{V}(T)$ be the quasivariety and variety generated by T, respectively. Since every subdirectly $\mathbf{N}$-irreducible lattice is a sublattice of T, we have that a class $\mathbf{N}_{si}$ of all subdirectly $\mathbf{N}$-irreducible lattices consists of the lattices $\mathbf{2}$, M_3, M_{3-3} and T (see Figures 1 and 2). It easy to see that M_3 is a unique non-distributive simple lattice in $\mathbf{N}_{si}$ and is a homomorphic image of T. Thus, the condition $a)$ of Tumanov's theorem is not valid for quasivarieties $\mathbf{N} \subset \mathbf{M}$.

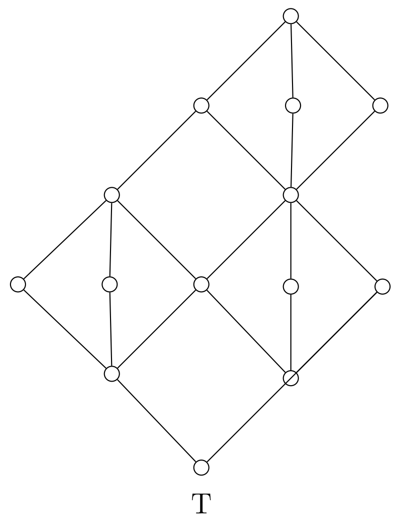

Figure 1 – **Lattice** T

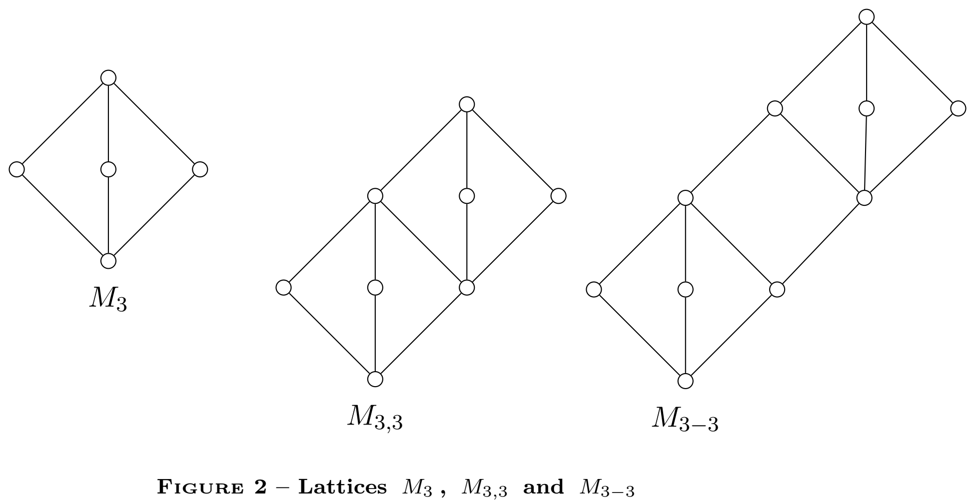

Figure 2 – **Lattices** M_3, $M_{3,3}$ **and** M_{3-3}

Let S be a non-empty subset of lattice L. Denote by $\langle S \rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 1$. $L_1 \cong M_{3-3}$ and $L_1 = \langle \{a_1, b_1, c_1, e, d\} \rangle$ (see Figure 3);

$n = 2$. L_2 is a modular lattice generated by $L_1 \cup \{a_2, b_2, c_2, d\}$ such that $b_1 = c_2$, $\langle \{a_2, b_2, c_2, e, b_1\} \rangle \cong M_3$, and $a_2 \vee b_2 = e \wedge d_1$, $d \vee b_1 = d_1$, and $b_2 < d$ (see Figure 3).

$n > 2$. L_n is a modular lattice generated by the set $\{a_i, b_i, c_i \mid i \leq n\} \cup \{e, d\}$ such that a_i is not comparable with a_j and b_k for all $j \neq i$ and $k \leq n$, $b_{i-1} = c_i$, $\langle \{a_i, b_i, c_i\} \rangle \cong M_3$ for all $i < n$, $b_i \vee d = d_i$ for all $i < n$, and $b_n < d$ (see Figure 4).

One can see that L_n is a subdirect product of the lattices L_{n-1} and M_3 for any $n > 2$.

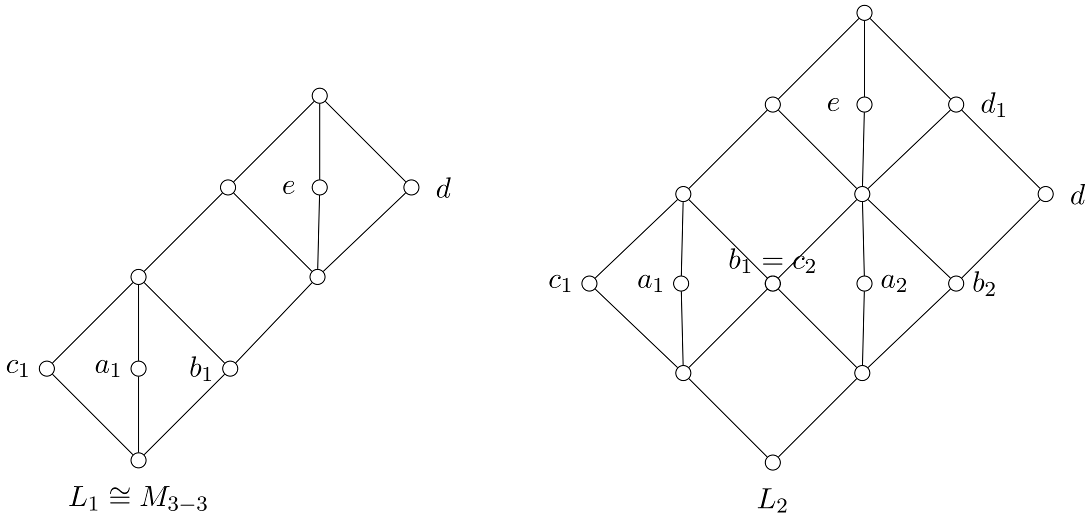

Figure 3 – **Lattices** L_1, L_2

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i \mid i \leq n\}$.

First, we prove two lemmas that will be used in proving the main result, Theorems 2 and 3.

Lemma 1. *For any $n > 1$ and a non-trivial congruence $\theta \in \mathrm{Con}(L_n)$ there is $1 < m < n$ such that $L_n/\theta \cong L_m$ or $L_n/\theta \cong M_{3,3}$ provided $(a_1, b_1) \notin \theta$, otherwise $L_n/\theta \cong L_m^-$.*

Proof of Lemma 1.

We prove by induction on $n > 2$. One can check that it is true for $n = 3$ because of $L_3/\theta \cong L_2$ or $L_3/\theta \cong M_{3,3}$ if $(a_1, b_1) \notin \theta$ and $L_3/\theta \cong L_2^-$ or $L_3/\theta \cong M_3$ for any non-trivial congruence $\theta \in \mathrm{Con}(L_3)$.

Let $n > 3$. And let u cover v in L_n and $\theta(u, v) \subseteq \theta$. By construction of L_n, we have $L_n/\theta(u, v) \cong L_{n-1}$ or $L_n/\theta(u, v) \cong L_{n-1}^-$.

Assume $(a_1, b_1) \notin \theta$. Since for every non-trivial congruence $\theta \in \mathrm{Con}(L_n)$ there are $u, v \in L_n$ such that u covers v and $\theta(u, v) \subseteq \theta$, we get

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)).$$

Since $L_n/\theta(u, v) \cong L_{n-1}$ we obtain

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)) \cong L_{n-1}/\theta',$$

for some $\theta' \in \mathrm{Con}(L_{n-1})$. And, by induction, $L_{n-1}/\theta' \cong L_m$ or $L_{n-1}/\theta' \cong M_{3,3}$ for some $m > 0$. Thus $L_n/\theta \cong L_m$ or $L_n/\theta \cong M_{3,3}$.

Now assume $(a_1, b_1) \in \theta$. Then $\theta(a_1, b_1) = \theta(u, v)$ and $L_n/\theta(u, v) \cong L_n^-$. Hence

$$L_n/\theta \cong (L_n/\theta(u, v))/(\theta/\theta(u, v)) \cong L_n^-/\theta',$$

for some $\theta' \in \mathrm{Con}(L_n^-)$. It is not difficult to check that $L_n^-/\theta' \cong L_m^-$ for some $m > 0$ (see Lemma 3.1 [13]). Thus $L_n/\theta \cong L_m$ or $L_n/\theta \cong L_m^-$.

Corollary 1. For all $n > 1$, there is no proper homomorphism from L_n to M_{3-3} and T.

Proof of Corollary 1.

We provide the proof for a proper homomorphism from L_n into M_{3-3}. It is not difficult to check that the same arguments hold for a proper homomorphism from L_n into T.

Assume $h : L_n \to M_{3-3}$, $n > 1$, is a proper homomorphism. Hence $\ker h$ is not a trivial congruence on L_n. By Lemma 1, $L_n/\ker h \cong L_m$ or $L_n/\theta \cong M_{3,3}$ or $L_n/\ker h \cong L_m^-$ for some $m > 1$. Thus $L_m = h(L_n) \leq M_{3-3}$. It is impossible because, by definition of L_m, $|L_m| > |M_{3-3}|$ for all $m > 1$, hence L_n is not a sublattice of M_{3-3}. Obviously, $M_{3,3}$ and L_M^- are not sublattices of M_{3-3}. Thus there is no such homomorphism h.

Lemma 2. *For every $n > 2$, a lattice L_n has the following properties:*
i) $L_n \leq_s L_{n-1} \times L_{n-1}$ *;*
ii) $L_n \in \mathbf{V}(M_{3,3}) = \mathbf{V}(T)$ *;*
iii) $L_n \notin \mathbf{Q}(T)$ *;*
iv) Every proper subalgebra of L_n belongs to $\mathbf{Q}(T)$.

Proof of Lemma 2.
i). One can check that $L_n/\theta(a_i, b_i) \cong L_{n-1}$ for all $1 < i \leq n$. Since $n > 2$ then $\theta(a_2, b_2), \theta(a_3, b_3) \in \mathrm{Con}(L_n)$ and $\theta(a_2, b_2) \cap \theta(a_3, b_3) = \Delta$. This means that $L_n \leq_s L_{n-1} \times L_{n-1}$.
ii). One can see that T is a subdirect product of M_3 and $M_{3,3}$. Hence $T \in \mathbf{V}(M_{3,3})$. On the other hand, by Jonsson lemma [14], every subdirectly irreducible lattice in $\mathbf{V}(T)$ is a homomorphic image of some sublattice of T. Hence $M_{3,3} \in \mathbf{V}(T)$. Thus $\mathbf{V}(M_{3,3}) = \mathbf{V}(T)$, and, by $i)$ and induction on n, we get $L_n \in \mathbf{V}(T)$.
iii). Suppose $L_n \in \mathbf{Q}(T)$ for some $n > 1$. Then L_n is a subdirect product of subdirectly $\mathbf{Q}(T)$-irreducible algebras. Since every subdirectly $\mathbf{Q}(T)$-irreducible algebra is a subalgebra of T, we get that L_n is a subdirect product of subalgebras of T. By Lemma 1, there is no proper homomorphism from L_n onto T or M_{3-3}. Hence $L_n \in \mathbf{Q}(M_3)$ for all $n > 1$. It is impossible because $M_{3-3} \leq L_n$ and $M_{3-3} \notin \mathbf{Q}(M_3)$.
iv). We prove by induction on n. It is true for $n \leq 2$ by manual checking. Let $n > 2$ and let S be a maximal sublattice of L_n. Since the lattice L_n is generated by the set of double irreducible elements $\{a_1, \ldots, a_n, c_1, e, d\}$, there is $0 < i \leq n$ such that $a_i \notin S$ or $c_1 \notin S$ or $e \notin S$ or $d \notin S$.
Suppose $c_1 \notin S$. One can see that $\langle S \rangle \leq_s \mathbf{2} \times M_3 \times L^-_{n-1}$. Since $L_{n-1} \leq_s M_3^{n-1}$ we get $\langle S \rangle \in \mathbf{Q}(M_3) \subset \mathbf{Q}(T)$.
Suppose $e \notin S$. Then $\langle S \rangle \leq_s \mathbf{2} \times L^-_n \leq_s \mathbf{2} \times M_3^n \in \mathbf{Q}(M_3) \subset \mathbf{Q}(T)$.
Suppose $d \notin S$. Put $S_m = \{\{a_1, \ldots, a_m, c_1, e\}$, $m < n$, and $T_m = \langle S_m \rangle$. One can see that $T_m/\theta(a_i, b_i) \cong T_{m-1}$ for all $1 < i < m$. And $T_m/\theta(a_1, b_1) \cong L^-_{m-1}$. Since $\theta(a_1, b_1) \cap \theta(a_i, b_i) = \Delta$, by distributivity of $\mathrm{Con}(T_m)$, we have $\theta(a_1, b_1) \cap (\bigvee\{\theta(a_i, b_i) \mid 1 < i < m\}) = \Delta$. Since $T_m/(\bigvee\{\theta(a_i, b_i) \mid 1 < i < m\}) \cong T$ we obtain $\langle S_m \rangle \leq_s T \times L^-_{n-1} \leq_s T \times M_3^{n-1} \in \mathbf{Q}(T)$.
Suppose $a_i \notin S$. Since $n > 1$ and S is a maximal sublattice, then there are $i \neq k \neq l \neq i$ such that $\theta(b_k, c_k), \theta(b_l, c_l) \in \mathrm{Con}(L_n)$,

$$\theta(b_k, c_k) \cap \theta(b_l, c_l) = \Delta.$$

and

$$L_n/\theta(b_k, c_k) \cong L_n/\theta(b_l, c_l) \cong L_{n-1} \quad \text{or} \quad \{L_n/\theta(b_k, c_k), L_n/\theta(b_l, c_l)\} = \{L_{n-1}, L^-_{n-1}\}.$$

We provide the proof for the first case, $L_n/\theta(b_k, c_k) \cong L_n/\theta(b_l, c_l) \cong L_{n-1}$. These isomorphisms mean that $L_n \leq_s L_{n-1} \times L_{n-1}$ and $S \leq L_{n-1} \times L_{n-1}$. Let $h_k : L_n \to L_{n-1}$ and $h_l : L_n \to L_{n-1}$ are homomorphisms such that $\ker h_k = \theta(b_k, c_k)$ and $\ker h_l = \theta(b_l, c_l)$. Since $(a_i, b_i) \notin \theta(b_k, c_k) \cup \theta(b_l, c_l)$ then $h_k(S)$, $h_l(S)$ are proper sublattices of L_{n-1}. And, by induction, $h_k(S), h_l(S) \in \mathbf{Q}(T)$. As $b_k, c_k, b_l, c_l \in S$, the restrictions of congruences $\theta(b_k, c_k)|_S$ and $\theta(b_l, c_l)|_S$ on the algebra S are not trivial congruences on S. Moreover $\theta(b_k, c_k)|_S \cap \theta(b_l, c_l)|_S = \Delta$. It means $S \leq_s h_k(S) \times h_l(S)$. Hence $S \in \mathbf{Q}(T)$. Since every maximal proper subalgebra of L_n belongs to $\mathbf{Q}(T)$ then every proper subalgebra of L_n belongs to $\mathbf{Q}(T)$.
It is not difficult to check that for $\{L_n/\theta(b_k, c_k), L_n/\theta(b_l, c_l)\} = \{L_{n-1}, L^-_{n-1}\}$ the same arguments hold.
For quasivariety $\mathbf{Q}(T)$ generated by the lattice T, the lattice L_n satisfies the conditions of the following folklore fact: A locally finite quasivariety $\mathbf{K}$ is not finitely axiomatizable if for any positive integer $n \in N$ there is a finite algebra L_n such that $L_n \notin \mathbf{K}$ and every n-generated subalgebra of L_n belongs to $\mathbf{K}$. Indeed, by Lemma 2(iii), $L_n \notin \mathbf{Q}(T)$ for all $n > 1$. Since L_n is generated by at least $n + 1$ double irreducible elements then every n-generated subalgebra of L_n is a proper subalgebra. By Lemma 2(iv), every n-generated subalgebra of L_n belongs to $\mathbf{Q}(T)$. Hence $\mathbf{Q}(T)$ has no finite basis of quasi-identities. Thus, we establish the following fact.

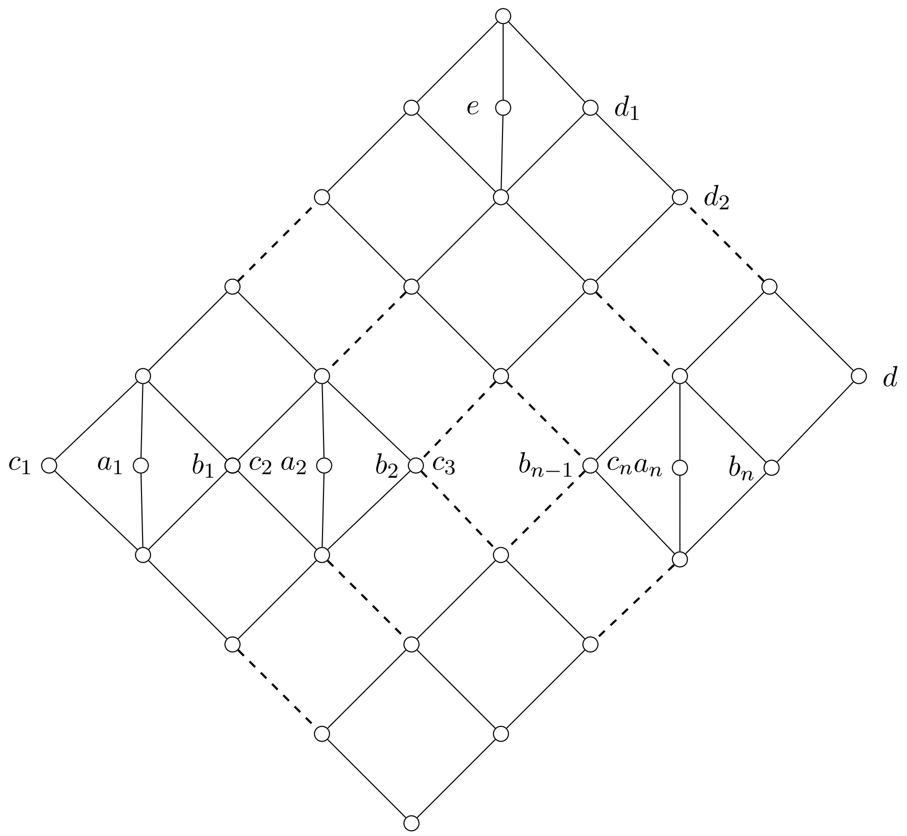

Figure 4 – Lattice L_n , $n \geq 2$

Theorem 2. *The quasivariety generated by the lattice* T *has no finite basis of quasi-identities.*

Now to prove that T generates a non-standard topological quasivariety, we will use the following lemma. It can be obtained from Lemma 3.3 of the paper [8]:

Lemma 3. *Let* $\mathbf{R}$ *be a quasivariety, and let* $A = \varprojlim\{A_n \mid n \in N\}$ *be a surjective inverse limit of finite algebras. Suppose that* $A \in \mathbf{R}$ *and there are* $a, b \in A$ *such that* $a \neq b$ *and* $\varphi(a) = \varphi(b)$ *for any homomorphism* $\varphi : A \to M$ *with* $M \in \mathbf{R}$ *and* M *is finite. Then* $\mathbf{R}$ *is not standard.*

The following theorem is true.

Theorem 3. *The topological quasivariety generated by the lattice* T *is not standard.*

Proof of Theorem 3.

So, to prove this statement, we need to check the feasibility of the conditions of Lemma 3.

Let $\varphi_{n,n-1}$ be a homomorphism from L_n to L_{n-1} such that $\ker \varphi_{n,n-1} = \theta(a_n, b_n)$, and $\varphi_{n,n}$ an identity map for all $n > 1$ and $m < n$. And let $\varphi_{n,m} = \varphi_{m+1,m} \circ \cdots \circ \varphi_{n,n-1}$. It can be seen that $\{L_n; \varphi_{n,m}, N\}$ forms inverse family, where N is the linear ordered set of positive integers.

We denote $L = \varprojlim\{L_n \mid n \in N\}$ and show that $L \in \mathbf{Q}(T)$.

Let α be a quasi-identity of the following form

$$\&_{i \leq r} p_i(x_0, \ldots, x_{n-1}) \approx q_i(x_0, \ldots, x_{n-1}) \to p(x_0, \ldots, x_{n-1}) \approx q(x_0, \ldots, x_{n-1}).$$

Assume that α is valid on $\mathbf{Q}(T)$ and

$$L \models p_i(a_0, \ldots, a_{n-1}) = q_i(a_0, \ldots, a_{n-1}) \quad \text{for all} \quad i < r,$$

for some $a_0, \ldots, a_{n-1} \in L$. From the definition of inverse limit we have that $L \leq_s \prod_{i \in I} L_i$. Therefore

$$L_s \models p_i(a_0(s), \ldots, a_{n-1}(s)) = q_i(a_0(s), \ldots, a_{n-1}(s)) \quad \text{for all} \quad i < r.$$

Each at most n spawned subalgebra of L_s belongs to $\mathbf{Q}(T)$ for all $s > n$, by Lemma 2(iv). Hence α is true in L_s for all $s > n$. And this in turn entails

$$L_s \models p(a_0(s), \ldots, a_{n-1}(s)) = q(a_0(s), \ldots, a_{n-1}(s)).$$

Since $a_i(m) = \varphi_{s,m}(a_i(s))$ for all $0 \leq i < n$ and $m < s$, we get

$$L_m \models p(a_0(m), \ldots, a_{n-1}(m)) = q(a_0(m), \ldots, a_{n-1}(m)) \quad \text{for all} \quad m < s.$$

So

$$L \models p(a_0, \ldots, a_{n-1}) = q(a_0, \ldots, a_{n-1}).$$

Hence $L \models \alpha$, for every α that is valid on $\mathbf{Q}(T)$. This proves that $L \in \mathbf{Q}(T)$.

We obtain $\varphi_{n,m}(a_1) = a_1$ and $\varphi_{n,m}(b_1) = b_1$, by definition of $\varphi_{n,n-1}$. And $a = (a_1, \ldots, a_1, \ldots)$, $b = (b_1, \ldots, b_1, \ldots) \in L$, by definition of inverse limit. Let $\varphi : L \to M$ be a homomorphism, $M \in \mathbf{Q}(T)$ and M finite. There is $n > 2$ and homomorphism $\psi_M : L_n \to M$ such that $\alpha = \varphi_n \circ \psi_M$ for some surjective homomorphism $\varphi_n : L \to L_n$ (by universal property of inverse limit). Since $\psi_M(L_n) \leq M \leq (T)^k$ for some $k > 0$, by Corollary 1 of Lemma 1, we obtain that $\psi_M(L_n)$ is trivial. That is $\psi_M(x) = 1$ for all $x \in L_n$. So we get $\alpha(a) = \alpha(b)$.

Thus, we obtain that the topological quasivariety generated by T is not standard.

We note that there is an infinite number of lattices similar to the lattice T.

The proof of Theorem 3 gives us the following more general result.

Theorem 4. *Let L be a finite lattice such that $M_{3,3} \not\leq L$, $T \leq L$ and $L_n \not\leq L$ for all $n > 1$. Then the topological quasivariety generated by the lattice L is not standard.*

Conclusion. In the present work we construct the finite modular lattice T that does not satisfy one of Tumanov's conditions but the quasivariety generated by this lattice is not finitely based. It has no finite basis of quasi-identities. We investigate the topological quasivariety generated by the constructed lattice and prove that it is not standard. And we would like to note that there is an infinite number of lattices similar to this lattice T.

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Белгілі бір ақырлы модулярлық торлардан пайда болған квазикөпбейнелердің кейбір қасиеттері туралы

Аннотация: Дискретті топологиялы ақырлы A алгебрасы оның бос емес декарттық дәрежелерінің сәйкес декарттық топологияларда тұйық барлық топологиялық тұйық ішкі алгебраларынан тұратын топологиялық квазикөпбейне тудырады. Егер Q(A)-де алгебралық редукторы бар әр бульдік топологиялық алгебра профинит болса, онда бұл топологиялық квазикөпбейне стандартты болып табылады. Мақалада Тумановтың бір шартын қанағаттандырмайтын, бірақ ол арқылы құрылған Q(T) квазикөпбейне ақырлы негіздік болмайтын T ақырлы модулярлы торы құрылады. T торы арқылы пайда болған топологиялық квазикөпбейне зерттеліп, оның стандартты емес екендігі дәлелденді. Сонымен қатар, T торына ұқсас шексіз торлар бар екенін атап өткіміз келеді.

Түйін сөздер: тор, ақырғы тор, модулярлық тор, Туманов шарттары, квазикөпбейне, топологиялық квазикөпбейне, стандартты топологиялық квазикөпбейне.

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О некоторых свойствах квазимногообразий, порожденных определенными конечными модулярными решетками

Аннотация: Конечная алгебра A с дискретной топологией порождает топологическое квазимногообразие, состоящее из всех топологически замкнутых подалгебр непустых декартовых степеней алгебры A, замкнутых в соответствующих декартовых топологиях. Это топологическое квазимногообразие является стандартным, если каждая булева топологическая алгебра с алгебраическим редуктом в Q(A) является профинитной. В статье проводится построение конечной модулярной решетки T, которая не удовлетворяет одному из условий Туманова, но квазимногообразие Q(T), порожденное этой решеткой, не является конечно базируемым. Исследуется топологическое квазимногообразие, порожденное решеткой T, и доказано, что оно не является стандартным. Также необходимо отметить, что существует бесконечное множество решеток, подобных решетке T.

Ключевые слова: решетка, конечная решетка, модулярная решетка, условия Туманова, квазимногообразие, топологическое квазимногообразие, стандартное топологическое квазимногообразие.

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