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Pinned point configurations and Hausdorff dimension [1]

Abstract: We prove that if the Hausdorff dimension of a compact subset E of $\mathbb{R}^d$ with $d \geq 2$ is sufficiently large, and if G is a star-like graph with two parts, and each of its parts is a rigid graph, then the Lebesgue measure in the appropriate dimension, of the set of distances in E specified by the graph is positive. We also prove that if $dim_{\mathcal{H}}(E)$ is sufficiently large, then

$$\int \nu_G(r\vec{t})d\nu_G(\vec{t}) > 0,$$

where ν_G is the measure on the space of distances specified by G which is induced by a Frostman measure. In particular, this means that for any $r > 0$ there exist many configurations encoded by $\vec{t}$ with vertices in E such that the vertices of $r\vec{t}$ are also in E.

Keywords: finite point configurations, group actions, simplexes, Hausdorff dimension.

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1. Introduction

Let G be a connected graph on $k+1$ vertices. Let $V = \{x^1, x^2, \ldots, x^{k+1}\}$ denote the vertex set and e_G the edge map, where $e_G(i,j) = 1$ if x^i and x^j are connected by an edge, and 0 otherwise. We will only consider undirected graphs with no self-edges, so $e_G(i,i) = 0$ and $e_G(i,j) = e_G(j,i)$ for all i, j. Let $\mathcal{E}(G)$ denote the edge set, namely

$$\left\{(i,j) \in V \times V : e(x^i, x^j) = 1\right\} / \sim,$$

where $\sim$ is the equivalence relation $(i,j) \sim (j,i)$.

Given such a graph and a compact subset of $\mathbb{R}^d$, we are interested in the set of various point-configurations specified by the graph. More precisely, given a Frostman measure on the compact set, we define the induced measure on the space of distances specified by the garph.

Definition 1. Let G be a graph as above, $E \subset \mathbb{R}^d$, $d \geq 2$ a compact set and μ a Frostman measure on E. Define the induced measure ν_G by the relation

$$\int f(\vec{t})d\nu_G(\vec{t}) = \int \ldots \int f(D_G(x^1, \ldots, x^{k+1}))d\mu(x^1)d\mu(x^2)\ldots d\mu(x^{k+1}),$$

where $\vec{t} = \{t_{ij}\}_{(i,j)\in\mathcal{E}(G)}$ is a set of positive real numbers, $D_G(x^1, \ldots, x^{k+1})$ is a vector of length equal to $\#\mathcal{E}(G)$ with entries $|x^i - x^j|$ for $(i,j) \in \mathcal{E}(G)$, with the entries ordered in the dictionary order.

Given such a compact set and a graph, for point configurations in the set we define their distance-profiles specified by the graph.

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Definition 2. Given a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, define

$$\Delta_G(E) = \left\{ D_G(x^1, \ldots, x^{k+1}) : x^j \in E \right\} \subset \mathbb{R}^{\#\mathcal{E}(G)}.$$

Also define

$$\Delta_G^r(E) = \left\{ \vec{t} \in \Delta_G(E) : r\vec{t} \in \Delta_G(E) \right\} \subset \Delta_G(E).$$

For $\epsilon > 0$, define a smooth approximation of ν_G on $\mathbb{R}^k$ by the density

$$\nu_G^\epsilon(\vec{t}) = \int \cdots \int \prod_{t_{ij} \in \mathcal{E}(G)} \sigma_{t_{ij}}^\epsilon (x^i - x^j) \; d\mu(x^1)...d\mu(x^{k+1})$$

$$= \int \cdots \int \prod_{i=1}^{n} T_{G_i}^\epsilon (x^i) \; d\mu(x^1)...d\mu(x^{k+1}),$$

where T_{G_i} encodes the part that belongs to G_i. Let $\sigma_{t_{ij}}$ be the normalized surface measure on the sphere of radius t_{ij} and $\sigma_{t_{ij}}^\epsilon(t) := \sigma_{t_{ij}} * \rho_\epsilon(t)$, with $\rho \in C_0^\infty(\mathbb{R}^d)$, $\rho \geq 0$, $supp(\rho) \subset \{|s| < 1\}$, $\int \rho = 1$, and $\rho_\epsilon(t) = \epsilon^{-d}\rho(\epsilon^{-1}t)$. Then each $\nu_G \in C_0^\infty$ and $\nu_G^\epsilon \to \nu_G$ weak* as $\epsilon \to 0$.

Thus,

$$\nu_G(\Delta_G^r(E)) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^k} \nu_G^\epsilon(r\vec{t}) \; d\nu_G(\vec{t}).$$

Definition 3. Let G be a graph that can be decomposed as follows. Let $G = \cup_i G_i$ where $G_1, \ldots, G_n$ is a family of connected graphs. Suppose that any G_i has exactly one vertex in common with any other G_j if $i \neq j$, and no other vertices in common between G_i and G_j, and there are no edges connecting vertices in G_i to vertices in G_j if $i \neq j$ except for their common point. Then we call G *a star* of G_i.

In this paper, we consider the case when all such G_i are rigid. A graph being rigid essentially means that continuous motion of the points of the configuration maintaining the edge length constraints comes from a family of distance-preserving Euclidean motions. The precise definition is the following.

Definition 4. Given a graph G with $V = \{x^1, x^2, \ldots, x^{k+1}\}$ being its vertex set, let K be the smallest graph containing G such that K is a complete graph. Let

$$F_G = \{|x^i - x^j|^2 : t_{ij} \textit{ is an edge of } G\}.$$

An infinitesimal motion of G is $\vec{u} = (u^1, ..., u^{k+1})$, a $(k+1)$-tuple $\vec{u}$ of vectors $u^j \in \mathbb{R}^d$ such that $DF_G \cdot \vec{u} = 0$.

If the set of infinitesimal motion of G and the set of infinitesimal motion of K are the same set, then G is called an *infinitesimal rigid graph.*

For a detailed discussion of rigidity in this sense refer to [2].

Our main results are the following.

Theorem 1. *Let G be a star of 2 graphs $\{G_i\}$ such that both G_i are infinitesimally rigid. For every i let $k_i + 1$ be the number of vertices G_i has and set $k = k_1 + k_2$, so that G has $k+1$ vertices. If $k \geq 4$, $d \geq 2$ and E is a compact subset of $\mathbb{R}^d$ of Hausdorff dimension larger than $\frac{dk-d+1}{k}$ then*

$$\mathcal{L}^m(\Delta_G(E)) > 0, \tag{1}$$

where m is the number of edges of G.

Note, by the definition of a rigid graph, we have that if $k_1 > d$, to compute the number of its edges, each of the vertices has d components, and we subtract the dimension of the Euclidean motion group. So the number of edges of G_1 is $d(k_1+1) - \binom{d+1}{2}$. If $k_1 \leq d$, it has to be a k_1- simplex, so the number of edges of G_1 is $\binom{k_1+1}{2}$. Similarly for G_2, if $k_2 > d$, the number of edges of G_2 is $d(k_2+1) - \binom{d+1}{2}$ and if $k_2 \leq d$, the number of edges of G_2 is $\binom{k_2+1}{2}$.

Therefore, if $k_1, k_2 > d$,

$$m = \sum_{i=1}^{2} \left[d(k_i+1) - \binom{d+1}{2} \right] = d(k+1) - 2\binom{d+1}{2} = dk - d^2.$$

If $k_1 > d$ and $k_2 \le d$,

$$m = d(k_1+1) - \binom{d+1}{2} + \binom{k_2+1}{2}.$$

If $k_1 \le d$ and $k_2 > d$,

$$m = d(k_2+1) - \binom{d+1}{2} + \binom{k_1+1}{2}.$$

If $k_1, k_2 \le d$,

$$m = \binom{k_1+1}{2} + \binom{k_2+1}{2}.$$

remark 1. Note, that the dimensional threshold we obtain is just the case $n = 2$. We expect that a similar result will be proved in the case for general n.

That said, the present result is still an improvement on currently available thresholds. Since the graph G in the above theorem is a subgraph of a $(k+1)-$ simplex, the results of [2] give that (1) holds when the Hausdorff dimension of E is larger than $d - \frac{1}{k+1}$. Since $d \ge 2$, our new bound $\frac{dk-d+1}{k}$ is an improvement. Also from [7], we know that (1) holds when the Hausdorff dimension is larger than $\frac{dk+1}{k+1}$. Since $\frac{dk+1}{k+1} > \frac{dk+1-d}{k}$ when $d \ge 2$, our threshold is an improvement on that as well.

In order to state our second result, we need the following definition.

Definition 5. Let $d \ge 2$, $k \ge 1$. Let G be a connected graph on $k+1$ vertices as above. Let E be a compact subset of $\mathbb{R}^d$, $d \ge 2$. Define

$$s_G = \inf \left\{ s : dim_{\mathcal{H}}(E) > s \Rightarrow \ \nu_G \text{ is absolutely continuous, and } \int \nu_G^2(\vec{t}) d\vec{t} < \infty \right\}.$$

We say s_G is the L^2 *-threshold* corresponding to the pair (G, E).

Allan Greenleaf, the first and second listed authors proved that if $G = K_{k+1}$, $k \le d$, and $E \subset \mathbb{R}^d$, $d \ge 2$, is a compact set of Hausdorff dimension larger than s_G, then

$$\nu_G(\Delta_G^r(E)) > 0 \tag{2}$$

for any $r > 0$. Roughly speaking, this means that for any $r > 0$ there exists a statistically correct number of pairs of k-dimensional simplexes that are similar to one another with the similarity ratio equal to r. The purpose of the second main result is to establish this type of a result for star-like graphs.

Our second main result is the following.

Theorem 2. *Let E be a compact subset of $\mathbb{R}^d$. Let G be a star of 2 infinitesimally rigid graphs $\{G_i\}$. Suppose that*

$$\int \nu_{G_i}(r\vec{t}) d\nu_{G_i}(\vec{t}) > 0, \tag{3}$$

and $dim_{\mathcal{H}}(E) > s_{G_i}$ for all i. Then, if $dim_{\mathcal{H}}(E) > s = \max\{s_{G_i}\}$, we have

$$\int \nu_G(r\vec{t}) d\nu_G(\vec{t}) > 0. \tag{4}$$

2. Proof of Theorem 1

We first prove the following proposition, which will help us to prove Theorem 1.

Let $O_d(\mathbb{R})$ be the orthogonal group of rotations of $\mathbb{R}^d$ and given $\theta \in O_d(\mathbb{R})$ define the measure λ_θ on $\mathbb{R}^d$ via the relation

$$\int f(x)d\lambda_\theta(x) = \int\int f(u-\theta v)d\mu(u)d\mu(v).$$

Proposition. *Let G be a star of n graphs $\{G_i\}$ such that all G_i are infinitesimally rigid. For every i let k_i+1 be the number of vertices G_i has and set $k=\prod_{i=1}^n k_i$, so that G has $k+1$ vertices. Then*

$$\int \nu_G^2(\vec{t})d\vec{t} < \infty$$

if and only if

$$\lim_{\epsilon\to 0^+}\int\cdots\int \lambda^\epsilon_{\theta_1}(x-\theta_1 x')^{k-n+1}\prod_{i=2}^n \lambda^\epsilon_{\theta_i}(x-\theta_i x')d\mu(x)d\mu(x')\prod_{i=1}^n d\theta_i < \infty,$$

where λ^ϵ denotes the convolution of λ with the approximation to the identity at level ϵ.

P r o o f. Let ν_G^ϵ denote the convolution of ν_G with the approximation to the identity at level ϵ. We'll prove the proposition by induction on the number of components n of the star graph G. First, suppose that $n=2$.

Using the same method as in Proposition 3.1 in [6], we can directly get

$$\liminf_{\epsilon\to 0}\int \nu_G^\epsilon(\vec{t})d\nu_G(\vec{t}) \approx \int\cdots\int \lambda^\epsilon_\theta(x-\theta x')^{k_1}\lambda^\epsilon_\phi(x-\phi x')^{k_2}d\mu(x)d\mu(x')d\theta d\phi \tag{5}$$

where x is the common vertex of G_1 and G_2, θ and ϕ correspond to the rotation of G_1 and G_2 respectively.

Here and thereafter, $X \lesssim Y$ means there exists a constant C such that $X \le CY$. The relation $X \gtrsim Y$ is defined similarly. In addition we write $X \approx Y$ if both $X \lesssim$ and $X \gtrsim Y$ hold.

Then by the Three Line Lemma, the right-hand side of (5) can be approximated as

$$\approx \int\cdots\int \lambda^\epsilon_\theta(x-\theta x')^{k-1}\lambda^\epsilon_\phi(x-\phi x')d\mu(x)d\mu(x')d\theta d\phi,$$

which corresponds to an infinitesimal rigid graph with k vertices with an extra edge added.

Therefore,

$$\liminf_{\epsilon\to 0}\int {\nu_G^\epsilon}^2(\vec{t})d\vec{t} < \infty$$

if and only if

$$\int\cdots\int \lambda^\epsilon_\theta(x-\theta x')^{k-1}\lambda^\epsilon_\phi(x-\phi x')d\mu(x)d\mu(x')d\theta d\phi < \infty.$$

For general n, using the same method when we are dealing with n=2, we can directly get

$$\liminf_{\epsilon\to 0}\int \nu_G^\epsilon(\vec{t})d\vec{t} \approx \int\cdots\int \lambda^\epsilon_{\theta_1}(x-\theta_1 x')^{k_1}\nu^\epsilon_{G'}(\vec{t'})d\mu(x)d\mu(x')d\theta_1 d\vec{t'} \tag{6}$$

where G' is the subgraph of G containing only $G_2,...,G_n$, and $\vec{t'}$ correspond to $\mathcal{E}(G')$, which is the edge set of G', and x is the common vertex of all G_i.

By the inductive hypothesis, (6) is

$$\approx \int\cdots\int \lambda^\epsilon_{\theta_1}(x-\theta_1 x')^{k_1}\lambda^\epsilon_{\theta_2}(x-\theta_2 x')^{\sum_{i=2}^{n-1}k_i-n+2}\lambda^\epsilon_{\theta_n}(x-\theta_n x')d\mu(x)d\mu(x')d\theta_1 d\theta_2 d\theta_n,$$

and applying the case $n = 2$, we get this is

$$\approx \int \cdots \int \lambda^{\epsilon}_{\theta_1}(x-\theta_1 x')^{k-n+1} \prod_{i=2}^{n} \lambda^{\epsilon}_{\theta_i}(x-\theta_i x')d\mu(x)d\mu(x') \prod_{i=1}^{n} d\theta_i,$$

finishing the proof of Proposition 2.

We're now ready to prove Theorem 1:

P r o o f. [Proof of Theorem 1]

By Proposition 2, we only need to estimate

$$\int \cdots \int \lambda^{\epsilon}_{\theta_1}(x-\theta_1 x')^{k-1} \lambda^{\epsilon}_{\theta_2}(x-\theta_2 x')d\mu(x)d\mu(x')d\theta_1 d\theta_2. \tag{7}$$

Since (7) corresponds to a graph which is a star of graphs with all except one of its components being a single edge, let's use t to denote the edge corresponding to λ_{θ_2} in this new graph. Then (7) is equal to

$$\int \cdots \int \lambda^{\epsilon}_{\theta_1}(x-\theta_1 x')^{k-1} \left(\sigma^{\epsilon}_t * \mu(x)\sigma^{\epsilon}_t * \mu(x')\right) d\mu(x)d\mu(x')d\theta_1 dt \tag{8}$$

Let κ_θ be defined similarly to λ_θ, via the relation

$$\int f(x)d\kappa_{\theta,t}(x) = \int\int f(u-\theta v)\cdot \sigma^{\epsilon}_t * \mu(u)\sigma^{\epsilon}_t * \mu(v)d\mu(u)d\mu(v).$$

Then by this definition, we get that (8) is equal to

$$\int \cdots \int \lambda^{\epsilon}_{\theta_1}(z)^{k-1}\kappa^{\epsilon}_{\theta_1,t}(z)dzd\theta_1 dt.$$

We use the Littlewood-Paley decomposition of it, and here the Littlewood-Paley piece is defined by $\hat{\lambda}_{\theta,j} = \hat{\lambda}_\theta(\xi)\rho(2^{-j}\xi)$, where ρ is a nonnegative bump function supported on $\{\frac{1}{2} \le ||\xi|| \le 2\}$, such that $\sum_j \rho_j(\xi) = 1$ for all ξ where $\rho_j(\xi) = \rho(2^{-j}\xi)$.

So we have that (7) is

$$= \sum_{j_0,j_1,\ldots,j_{k-1}} \int \cdots \int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)\kappa^{\epsilon}_{\theta_1,t,j_0}(z)dzd\theta_1 dt$$

$$\approx \sum_{j_0}\sum_{0\le j_1\le\cdots\le j_{k-1}} \int \cdots \int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)\kappa^{\epsilon}_{\theta_1,t,j_0}(z)dzd\theta_1 dt \tag{9}$$

$$\le \sum_{j_0}\sum_{0\le j_1\le\cdots\le j_{k-1}} \int \cdots \int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)||\kappa^{\epsilon}_{\theta_1,t,j_0}(z)||_\infty dzd\theta_1 dt.$$

And we have

$$||\kappa^{\epsilon}_{\theta_1,t,j}||_\infty \lesssim ||\beta_j||^2_{L^2(\mu)} \tag{10}$$

where $d\beta(x) = \sigma^{\epsilon}_t * \mu(x)d\mu(x)$.

Let ψ be a smooth positive function such that $\psi \ge \hat{\rho}$ and $||\psi||$ is bounded. Such ψ exists because $|\hat{\rho}(x)| \le C_N(1+|x|)^N$ for some constant C_N and integer N. Then

$$||\beta_j||^2 \approx \int |\hat{\beta}_j(\epsilon)|^2 d\epsilon \approx \int |\hat{\beta}_j(\epsilon)|^2 \hat{\psi}(\frac{\epsilon}{2^j})d\epsilon$$

$$\approx 2^{dj}\int \cdots \int \psi(2^j(x-x'))\sigma^{\epsilon}_t * \mu(x)\sigma_t * \mu(x')d\mu(x)d\mu(x')$$

$$\lesssim 2^{j(d-s)}||\sigma^{\epsilon}_t * \mu||^2_{L^2(\mu)}.$$

According to Theorem 2.1 in [1], we have that $||\sigma^{\epsilon}_t * \mu||_{L^2(\mu)}$ is bounded when $s > \frac{d+1}{2}$. From the assumption we have $k \ge 4 > 2$ and $d \ge 2 > 1$. Then there is $(d-1)(k-2) > 0$, and we get $\frac{dk-d+1}{k} > \frac{d+1}{2}$, so the result from [1] applies and for each i, the left-hand side of (10) is bounded by $2^{j_0(d-s)}$. Therefore, each j_0 -th piece of (9) is

$$\lesssim 2^{j_0(d-s)} \sum_{0\le j_1\le\cdots\le j_{k-1}} \int\cdots\int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)dzd\theta_1.$$

Using the Plancherel theorem, we estimate this by

$$\lesssim 2^{j_0(d-s)(1)} \sum_{0\le j_1\le\cdots\le j_{k-1}} \int\cdots\int \hat{\lambda}_{\theta_1,j_1}(z)*\cdots*\hat{\lambda}_{\theta_1,j_{k-3}}(z)*\hat{\lambda}_{\theta_1,j_{k-1}}(z)\cdot\hat{\lambda}_{\theta_1,j_{k-2}}(z)dzd\theta.$$

The support of $\hat{\lambda}_{\theta_1,j_1}*\cdots*\hat{\lambda}_{\theta_1,j_{k-3}}*\hat{\lambda}_{\theta_1,j_{k-1}}$ has scale $2^{j_1}+\cdots+2^{j_{k-3}}+2^{j_{k-1}}\sim 2^{j_{k-1}} > 2^{j_{k-1}-1}$, and the support of $\hat{\lambda}_{\theta_1,j_{k-2}}$ has scale $2^{j_{k-2}}$. Therefore, if $j_{k-1}-j_{k-3}\ge 2$, then $2^{j_{k-1}-1} > 2^{j_{k-2}}$ and

$$\int\cdots\int \hat{\lambda}_{\theta_1,j_1}(z)*\cdots*\hat{\lambda}_{\theta_1,j_{k-3}}(z)*\hat{\lambda}_{\theta_1,j_{k-1}}(z)\cdot\hat{\lambda}_{\theta_1,j_{k-2}}(z)dzd\theta_1 = 0$$

in this case.

If $j_{k-1}-j_{k-2}=1$, then by Cauchy-Schwarz

$$\left(\int\cdots\int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)dzd\theta_1\right)^2$$
$$\le \left(\int\cdots\int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-3}}(z)\left(\lambda^{\epsilon}_{\theta_1,j_{k-1}}(z)\right)^2 dzd\theta_1\right)$$
$$\cdot\left(\int\cdots\int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-3}}(z)\left(\lambda^{\epsilon}_{\theta,j_{k-2}}(z)\right)^2 dzd\theta_1\right)$$

which reduces to the product of two integral with their largest two indices for λ equal. It follows that we only need to consider the case when $j_{k-1}=j_{k-2}$. Similarly, by the orthogonal property of Littlewood-Paley pieces, we only need to consider the case $j_0=j_{k-1}=j_{k-2}=j$

Thus, using (10), we have that (9) is

$$\lesssim 2^{j(d-s)} \sum_{0\le j_1\le j_2\le\cdots\le j_{k-3}\le j} \int\cdots\int \lambda^{\epsilon}_{\theta_1,j_1}(z)\ldots\lambda^{\epsilon}_{\theta_1,j_{k-3}}(z)\left(\lambda^{\epsilon}_{\theta_1,j}(z)\right)^2 dzd\theta_1$$
$$\lesssim 2^{j(d-s)}\sum_j \sum_{0\le j_1\le j_2\le\cdots\le j_{k-3}\le j} 2^{(j_1+\cdots+j_{k-3})(d-s)} \int\cdots\int \left(\lambda^{\epsilon}_{\theta_1,j}(z)\right)^2 dzd\theta_1$$
$$\le 2^{j(d-s)}\cdot C\sum_j 2^{j(k-3)(d-s)} \iint \left(\lambda^{\epsilon}_{\theta_1,j}(z)\right)^2 dzd\theta_1$$

By Section 5 and Theorem 3.1 in [7], we can use the Wolff-Erdogan Theorem to get the following result:

$$\int\cdots\int \left(\lambda^{\epsilon}_{\theta_1,j}(x-\theta_1 x')\right)^2 d\mu(x)d\mu(x')d\theta_1 \lesssim 2^{j(d-s)-j\gamma(s,d)}$$

where $\gamma(s,d)=s-1$ if $s\ge\frac{d+2}{2}$, and $\gamma(s,d)=\frac{d+2s-2}{4}$ if $\frac{d}{2}\le s\le\frac{d+2}{2}$.

It follows that (9) is

$$\lesssim \sum_j 2^{j(d-s)}2^{j(k-3)(d-s)}2^{j(d-s)}2^{-j\gamma(s,d)} = \sum_j 2^{j[(k-1)(d-s)-\gamma(s,d)]}.$$

If $k\ge 4$ and $d>2$ are true, then a simple computation shows that $\frac{dk-d+1}{k}\ge\frac{d+2}{2}$. Thus if $s>\frac{dk-d+1}{k}$, then $s>\frac{d+2}{2}$, which implies that $(k-1)(d-s)-\gamma(s,d)=(k-1)(d-s)-(s-1)<0$.

If $k\ge 4$ and $d=2$ are true, then $s>\frac{dk-d+1}{k}=\frac{2k-1}{k}>1=\frac{d}{2}$, which implies that $(k-1)(d-s)-\gamma(s,d)=(k-1)(2-s)-\frac{2+2s-2}{4}=2k-2-(k-\frac{1}{2})s$. Simple computation shows that we have $s>\frac{2k-1}{k}>\frac{2k-2}{k-\frac{1}{2}}$, which entails that $(k-1)(d-s)-\gamma(s,d)<0$.

3. Proof of Theorem 2

For $i = 1, 2$, let θ_i be rotations such that

$$r\theta_i(x^{j_1} - x^{j_2}) \in B(y^{j_1} - y^{j_2}, \epsilon)$$

for $t_{j_1 j_2}$ in G_i.

Suppose $r > 0$. Then we have

$$\int \nu_G^\epsilon(r\vec{t})\ d\nu_G(\vec{t}) = \int T_{G_1}^\epsilon(x) T_{G_2}^\epsilon(x)\ d\mu(x^1)...d\mu(x^{k+1})$$

$$\approx \epsilon^{-\binom{k_1}{2}-\binom{k_2}{2}} \underset{\substack{||y^i-y^j|-r|x^i-x^j||<\epsilon \\ for\ all\ i,j\ s.t.\ t_{ij}\in\mathcal{E}(G)}}{\int \cdots \int} \prod_{s=1}^{k+1} (d\mu(x^s)d\mu(y^s)).$$

For rotation θ_i, just like in the last section, λ_{r,θ_i} is defined to be a measure on $\mathbb{R}^d$ by

$$\int f(z)\ d\lambda_{r,\theta_i}(z) = \iint f(u - r\theta_i v)\ d\mu(u)d\mu(v)\ ,\ f \in C_0(\mathbb{R}^d).$$

It has total mass $||\lambda_{r,\theta_i}|| = \mu(E)^2$. Let $d\theta$ be the Haar probability measure on $O_d(\mathbb{R})$.

We have

$$\liminf_{\epsilon\to 0} \int \nu_G^\epsilon(r\vec{t})\ d\nu_G(\vec{t})$$

$$\approx \int \cdots \int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{k_1} \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2}\ d\mu(x)d\mu(y)d\theta_1 d\theta_2$$

$$= \int \cdots \int \left(\int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{k_1} d\theta_2\right) \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y).$$

Without loss of generality, we can assume $k_1 \geq k_2$.

By Cauchy-Schwarz, if k_1 is odd, then

$$\left(\int \cdots \int \left(\int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{k_1} d\theta_1\right) \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)$$

$$\cdot \left(\int \cdots \int \left(\int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right) d\theta_1\right) \cdot \int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\ d\mu(x)d\mu(y)\right)$$

$$\geq \left(\int \cdots \int \int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{\frac{k_1+1}{2}} d\theta_1 \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)^2.$$

Note, that the second term of the left-hand side of the above inequality corresponds to a star-like graph with 2 parts, so is bounded above by following exactly the same steps when we proving Theorem1 until the last step of that proof. Therefore,

$$\left(\int \cdots \int \left(\int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{k_1} d\theta_1\right) \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)$$

$$\gtrsim \left(\int \cdots \int \int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{\frac{k_1+1}{2}} d\theta_1 \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)^2.$$

If k_1 is even, we have

$$\left(\int \cdots \int \left(\int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{k_1} d\theta_1\right) \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)$$

$$\cdot \left(\int \cdots \int \int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\ d\mu(x)d\mu(y)\right)$$

$$\geq \left(\int \cdots \int \int \left(\lambda_{r,\theta_1}^\epsilon(y - r\theta_1 x)\right)^{\frac{k_1}{2}} d\theta_1 \left(\int \left(\lambda_{r,\theta_2}^\epsilon(y - r\theta_2 x)\right)^{k_2} d\theta_2\right)\ d\mu(x)d\mu(y)\right)^2.$$

Again, the second term of the left-hand side of above inequality corresponds to a star-like graph with 2 parts, so is bounded above because of the same reason in the odd case. Therefore,

$$\left(\int\cdots\int\left(\int\left(\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_1x)\right)^{k_1}d\theta_1\right)\left(\int\left(\lambda^{\epsilon}_{r,\theta_2}(y-r\theta_2x)\right)^{k_2}d\theta_1\right)\,d\mu(x)d\mu(y)\right)$$

$$\gtrsim\left(\int\cdots\int\int\left(\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_1x)\right)^{\frac{k_1}{2}}d\theta_1\left(\int\left(\lambda^{\epsilon}_{r,\theta_2}(y-r\theta_2x)\right)^{k_2}d\theta_2\right)\,d\mu(x)d\mu(y)\right)^2.$$

Using the above process repeatedly, we get

$$\int\cdots\int\left(\int\left(\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_ix)\right)^{k_1}d\theta_1\right)\left(\int\left(\lambda^{\epsilon}_{r,\theta_2}(y-r\theta_2x)\right)^{k_2}d\theta_2\right)\,d\mu(x)d\mu(y)$$

$$\gtrsim\left(\int\cdots\int\left(\int\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_1x)d\theta_1\right)\left(\int\left(\lambda^{\epsilon}_{r,\theta_2}(y-r\theta_2x)\right)d\theta_2\right)\,d\mu(x)d\mu(y)\right)^{2^m} \tag{11}$$

$$=\left(\int\cdots\int\left(\int\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_1x)d\theta_1\right)^2\,d\mu(x)d\mu(y)\right)^{2^m}$$

for some integer m, where m is the number of doing the above process. By Cauchy-Schwarz, this is

$$\geq\left(\int\cdots\int\lambda^{\epsilon}_{r,\theta_1}(y-r\theta_1x)d\theta_1\,d\mu(x)d\mu(y)\right)^{2^{m+1}}.$$

$$=\left(\int\cdots\int\lambda^{\epsilon}_{r,\theta_1}(z)d\theta_1\,dz\right)^{2^{m+1}}=\mu(E)^{2^{m+1}}$$

where E is a 2-chain.

Therefore for all $r>0$, (4) holds. This completes the proof.

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Жабық нүкте конфигурациялары және Хаусдорф өлшемілігі

Аннотация: Мақалада $d(d \geq 2)$ өлшемді R^d жиынының компактты E жиыншасының хаусдорфтік өлшемділігі жетерліктей үлкен және G - әрбір бөлігі қатаң граф болатын екі бөлікті жұлдызды граф болғанда, граф арқылы берілген E -дегі қашықтықтар жиынының сәйкес өлшемділікті Лебег өлшемі оң болатыны дәлелденді. Сонымен қатар, $dim_H(E)$ жетерліктей үлкен болғанда

$$\int \nu_G(r\vec{t})d\nu_G(\vec{t}) > 0$$

теңсіздігі орындалатыны дәлелденді. Мұндағы ν_G – G -де анықталған қашықтықтар кеңістігіндегі Фростмен өлшемі арқылы индукцияланған өлшем. Дербес жағдайда, бұл дегеніміз кез келген $r > 0$ үшін $r\vec{t}$ төбелері де E жататын $(\vec{t})$ кодталған, төбелері де E -де жататын конфигурациялар жиыны табылады.
Түйін сөздер: ақырлы нүктелі конфигурациялар, топтық амалдар, симплекстер, Хаусдорф өлшемілігі.

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Конфигурации закрытой точки и Хаусдорфова размерность

Abstract: В статье доказывается, что если хаусдорова размерность компактного E подмножества R^d с размерностью $d \geq 2$ достаточно велика, и если G есть звездный граф с двумя частями и каждая из его частей является жестким графом, то мера Лебега в соответствующей размерности набор расстояний в E , заданный графом, является положительной. Также доказано, что если $dim_H(E)$ является достаточно велико, то

$$\int \nu_G(r\vec{t})d\nu_G(\vec{t}) > 0,$$

где ν_G есть мера на пространстве расстояний, заданном G, которая индуцирована мерой Фростмена. В частности, это означает, что для любого $r > 0$ существует множество конфигураций, закодированных $(\vec{t}) >$ с вершинами в E , так что вершины $r\vec{t}$ также находятся в E .
Keywords: конечноточечные конфигурации, групповые действия, симплексы, хаусдорфова размерность.

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