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# On the Reduction of the Linear System of the Differential Equations with coefficients of oscillating type to the Triangular Kind in the Non-resonant Case

**Abstract:** For the linear homogeneous differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, the conditions of the existence of the transformation which leads it to triangular kind, are obtained in the non-resonant cases.

**Keywords:** linear differential systems, Fourier series.

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**Introduction.** In the theory of linear systems of differential equations is well known problem of the consruction for the linear homogeneous system of the differential equations

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where $x = \mathrm{colon}(x_1, ..., x_n)$, $A(t) = (a_{jk}(t))_{j,k=\overline{1,n}}$, Lyapunov's transformation

$$x = L(t)y,$$

which leads the system (1) to the triangular kind

$$\frac{dy}{dt} = T(t)y,$$

where $T(t) = (b_{jk}(t))_{j,k=\overline{1,n}}$, $b_{jk}(t) \equiv 0 \quad (j < k)$ [1–4].

In this paper, we assume, that the system (1) already reduced to a kind, close to triangular:

$$\frac{dx}{dt} = (T(t) + \mu P(t))x, \tag{2}$$

where $\mu$ – small parameter, and the matrix $P(t)$ has a some special kind. And we study the problem on bringing the system (2) to a purely triangular form

$$\frac{dy}{dt} = D(t)y,$$

where $D(t) = (d_{jk}(t))_{j,k=\overline{1,n}}$, $d_{jk} \equiv 0 \quad (j < k)$.

**Basic notations and definitions.**

Let $G(\varepsilon_0) = \{t, \varepsilon: \ 0 < \varepsilon < \varepsilon_0, \ t \in \mathbb{R}\}$.

**Definition 1.** We say, that a function $p(t, \varepsilon)$ belongs to a class $S(m; \varepsilon_0)$ ($m \in \mathbb{N} \cup \{0\}$), if

1) $p : G(\varepsilon_0) \to \mathbb{C}$, 2) $p(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect $t$;

3) $d^k p(t, \varepsilon)/dt^k = \varepsilon^k p_k^*(t, \varepsilon) \quad (0 \le k \le m)$,

$$\|p\|_{S(m;\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k^*(t, \varepsilon)| < +\infty.$$

Under the slowly varying function we mean the function of the class $S(m; \varepsilon_0)$.

**Definition 2.** We say, that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to a class $F(m;\varepsilon_0;\theta)$ $(m \in \mathbb{N}\cup\{0\})$, if this function can be represented as:

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

and:
1) $f_n(t,\varepsilon) \in S(m;\varepsilon_0)$;
2)

$$\|f\|_{F(m;\varepsilon_0;\theta)} \stackrel{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty,$$

3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\varphi(t,\varepsilon) \in \mathbb{R}^+$, $\varphi(t,\varepsilon) \in S(m;\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0$.

State some properties of the functions of the classes $S(m;\varepsilon_0)$, $F_0(m;\varepsilon_0;\theta)$ (the proofs are given in [5]). Let $k = \text{const}$, $p,q \in S(m;\varepsilon_0)$, $u,v \in F(m;\varepsilon_0;\theta)$. Then $kp$, $p \pm q$, $pq$ belongs to the class $S(m;\varepsilon_0)$, $ku$, $u \pm v$, $uv$ belongs to the class $F_0(m;\varepsilon_0;\theta)$, and
1) $\|kp\|_{S(m;\varepsilon_0)} = |k|\cdot\|p\|_{S(m;\varepsilon_0)}$.
2) $\|p \pm q\|_{S(m;\varepsilon_0)} \le \|p\|_{S(m;\varepsilon_0)} + \|q\|_{S_0(m,\varepsilon_0)}$.
3) $\|pq\|_{S(m;\varepsilon_0)} \le 2^m\|p\|_{S(m;\varepsilon_0)}\|q\|_{S_0(m;\varepsilon_0)}$.
4) $\|ku\|_{F(m;\varepsilon_0;\theta)} = |k|\cdot\|u\|_{F(m;\varepsilon_0;\theta)}$.
5) $\|u \pm v\|_{F(m;\varepsilon_0;\theta)} \le \|u\|_{F(m;\varepsilon_0;\theta)} + \|v\|_{F(m;\varepsilon_0;\theta)}$.
6) $\|uv\|_{F(m;\varepsilon_0;\theta)} \le 2^m\|u\|_{F(m;\varepsilon_0;\theta)}\cdot\|v\|_{F(m;\varepsilon_0;\theta)}$.

For $f(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$ we denote:

$$\Gamma_n[f] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\exp(-in\theta)d\theta \quad (n \in \mathbb{Z}).$$

In particular

$$\Gamma_0[f] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)d\theta.$$

**Definition 3.** For the vector $u = \text{colon}(u_1,\ldots,u_n)$ with elements from the class $F(m;\varepsilon_0;\theta)$ we define the norm:

$$\|u\|^*_{F(m;\varepsilon_0;\theta)} = \sum_{k=1}^{n} \|u_k\|_{F(m;\varepsilon_0;\theta)}.$$

**Statement of the Problem.** We consider the next system of differential equations:

$$\frac{dx}{dt} = (B(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x, \tag{3}$$

where $x = \text{colon}(x_1,...,x_n)$, $B(t,\varepsilon)$ – lower triangular matrix with the elements from $S(m;\varepsilon_0)$, and $P(t,\varepsilon,\theta) = (p_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,n}}$, $p_{jk}(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,n})$, $\mu \in (0,\mu_0)$ – the real parameter.

We study the problem of the existence of a transformation of kind

$$x = (E_n + \mu\Psi(t,\varepsilon,\theta,\mu))y, \tag{4}$$

$y = \text{colon}(y_1,...,y_n)$, $E_n$ – unit matrix of order $n$, $\Psi$ – matrix with elements from $F(l;\varepsilon_1;\theta)$ $(0 < l_1 \le m,\ 0 < \varepsilon_1 < \varepsilon_0)$, which leads at sufficiently small $\mu$ the system (3) to the kind:

$$\frac{dy}{dt} = K(t,\varepsilon,\theta,\mu))y, \tag{5}$$

where $K = (k_{jk}(t,\varepsilon,\theta,\mu))_{j,k=\overline{1,n}}$, $k_{jk} \equiv 0$ $(j < k)$, $k_{jk}(t,\varepsilon,\theta,\mu) \in F(l;\varepsilon_1;\theta)$.

We will study this problem for a third-order system $(n = 3)$ so as not to clutter up the presentation with secondary technical difficulties associated with the dimension of the system. All fundamental difficulties take place in this case too.

So, consider the system of the differential equations:

$$\frac{dx}{dt} = (B(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x, \tag{6}$$

$x = \operatorname{colon}(x_1, x_2, x_3)$,

$$B(t,\varepsilon) = \begin{pmatrix} b_{11}(t,\varepsilon) & 0 & 0 \\ b_{21}(t,\varepsilon) & b_{22}(t,\varepsilon) & 0 \\ b_{31}(t,\varepsilon) & b_{32}(t,\varepsilon) & b_{33}(t,\varepsilon) \end{pmatrix},$$

$b_{jk}(t,\varepsilon) \in S(m;\varepsilon_0)$ $(j,k = 1,2,3; j \geq k)$, $P(t,\varepsilon,\theta) = (p_{jk}(t,\varepsilon,\theta))_{j,k=1,2,3}$, $p_{jk}(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$.

**Auxiliary results.**

**Lemma 1.** *Let we have the system*

$$\frac{dv}{dt} = \left( A(t,\varepsilon) + \sum_{l=1}^{q} Q_l(t,\varepsilon,\theta)\mu^l \right) v, \tag{7}$$

$x = \operatorname{colon}(x_1, x_2, x_3)$, $q \in \mathbb{N}$,

$$A(t,\varepsilon) = \begin{pmatrix} im_{12}(t,\varepsilon) & -c_{32}(t,\varepsilon) & 0 \\ 0 & im_{13}(t,\varepsilon) & 0 \\ 0 & c_{21}(t,\varepsilon) & im_{23}(t,\varepsilon) \end{pmatrix}$$

$m_{jk}(t,\varepsilon) \in S(m;\varepsilon_0)$, $m_{jk}(t,\varepsilon) \in \mathbb{R}$, $c_{jk}(t,\varepsilon) \in S(m;\varepsilon_0)$, *and*

$$\begin{aligned} &\inf_{G(\varepsilon_0)} |m_{13}(t,\varepsilon) - m_{12}(t,\varepsilon) - n\varphi(t,\varepsilon)| \geq \gamma > 0, \\ &\inf_{G(\varepsilon_0)} |m_{23}(t,\varepsilon) - m_{12}(t,\varepsilon) - n\varphi(t,\varepsilon)| \geq \gamma > 0, \\ &\inf_{G(\varepsilon_0)} |m_{13}(t,\varepsilon) - m_{23}(t,\varepsilon) - n\varphi(t,\varepsilon)| \geq \gamma > 0, \end{aligned} \tag{8}$$

$n \in \mathbb{Z}$, $\varphi(t,\varepsilon)$ *– the function in the definition of class* $F(m;\varepsilon_0;\theta)$*, the elements of matrices* $Q_l$ $(l = \overline{1,q})$ *belongs to the class* $F(m;\varepsilon_0;\theta)$.

*Then there exists* $\mu_1 \in (0,\mu_0)$*, such that for all* $\mu \in (0,\mu_1)$ *there exists the Lyapunov's transformation of kind*

$$v = \left( E + \sum_{l=1}^{q} \Psi_l(t,\varepsilon,\theta)\mu^l \right) w, \tag{9}$$

*where elemens of matrices* $\Psi_l(t,\varepsilon,\theta)$ $(l = \overline{1,q})$ *belongs to the class* $F(m;\varepsilon_0;\theta)$*, which leads the system (7) to kind:*

$$\frac{dw}{dt} = \left( A(t,\varepsilon) + \sum_{l=1}^{q} U_l(t,\varepsilon)\mu^l + \varepsilon \sum_{l=1}^{q} V_l(t,\varepsilon,\theta)\mu^l + \mu^{q+1} W(t,\varepsilon,\theta,\mu) \right) w, \tag{10}$$

*where* $U_l(t,\varepsilon)$ *– the matrices with elements from* $S(m;\varepsilon_0)$*,* $V_l, W$ *– the matrices with elements from* $F(m-1;\varepsilon_0;\theta)$.

**Proof.** We substitute the expression (9) into system (7), and require that the transformed system has the kind (10). We obtain the next chain of matrix differential equations for detemining matrices $\Psi_1, ..., \Psi_q$:

$$\frac{d\Psi_1}{dt} = A(t,\varepsilon)\Psi_1 - \Psi_1 A(t,\varepsilon) + Q_1(t,\varepsilon,\theta) - U_1(t,\varepsilon) - \varepsilon V_1(t,\varepsilon,\theta), \tag{11}$$

$$\frac{d\Psi_l}{dt} = A(t,\varepsilon)\Psi_l - \Psi_l A(t,\varepsilon) + Q_l(t,\varepsilon,\theta) - \sum_{\nu=1}^{l-1} Q_\nu \Psi_{l-\nu} -$$

$$- \sum_{\nu=1}^{l-1} \Psi_\nu U_{l-\nu}(t,\varepsilon) - \varepsilon \sum_{\nu=1}^{l-1} \Psi_\nu V_{l-\nu}(t,\varepsilon,\theta) - U_l(t,\varepsilon) - \varepsilon V_l(t,\varepsilon,\theta), \; l = \overline{2,q}. \tag{12}$$

where $\Psi_l = (\psi_{jk}^l)_{j,k=1,2,3}$, $Q_l = (q_{jk}^l)_{j,k=1,2,3}$, $U_l = (u_{jk}^l)_{j,k=1,2,3}$, $V_l = (v_{jk}^l)_{j,k=1,2,3}$ $(l = \overline{1,q})$.

Then the matrix $W$ at sufficiently small values $\mu$ is determined from the equation:

$$\left(E+\sum_{l=1}^{q}\Psi_l\mu^l\right)W=\sum_{s=0}^{q-1}\left[\sum_{\sigma+\delta=s+q+1}(Q_\sigma\Psi_\delta-\Psi_\sigma U_\delta\right]\mu^s-$$

$$-\sum_{s=0}^{q-1}\left(\sum_{\sigma+\delta=s+q+1}\Psi_\sigma V_\delta\right)\mu^s. \qquad (13)$$

We consider the equation (11). In the component it looks like this:

$$\frac{d\psi_{11}^1}{dt}=-c_{32}(t,\varepsilon)\psi_{21}^1+q_{11}^1(t,\varepsilon,\theta)-u_{11}^1(t,\varepsilon)-\varepsilon v_{11}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{12}^1}{dt}=i(m_{12}(t,\varepsilon)-m_{13}(t,\varepsilon))\psi_{12}^1-c_{32}(t,\varepsilon)(\psi_{22}^1-\psi_{11}^1)-c_{21}(t,\varepsilon)\psi_{13}^1+$$

$$+q_{12}^1(t,\varepsilon,\theta)-u_{12}^1(t,\varepsilon)-\varepsilon v_{12}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{13}^1}{dt}=i(m_{12}(t,\varepsilon)-m_{23}(t,\varepsilon))\psi_{13}^1-c_{32}(t,\varepsilon)\psi_{23}^1+$$

$$+q_{13}^1(t,\varepsilon,\theta)-u_{13}^1(t,\varepsilon)-\varepsilon v_{13}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{21}^1}{dt}=i(m_{13}(t,\varepsilon)-m_{12}(t,\varepsilon))\psi_{21}^1+q_{21}^1(t,\varepsilon,\theta)-u_{21}^1(t,\varepsilon)-\varepsilon v_{21}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{22}^1}{dt}=c_{32}(t,\varepsilon)\psi_{21}^1-c_{21}(t,\varepsilon)\psi_{32}^1+q_{22}^1(t,\varepsilon,\theta)-u_{11}^1(t,\varepsilon)-\varepsilon v_{22}^1(t,\varepsilon,\theta), \qquad (14)$$

$$\frac{d\psi_{23}^1}{dt}=i(m_{13}(t,\varepsilon)-m_{23}(t,\varepsilon))\psi_{23}^1+q_{23}^1(t,\varepsilon,\theta)-u_{23}^1(t,\varepsilon)-\varepsilon v_{23}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{31}^1}{dt}=i(m_{23}(t,\varepsilon)-m_{12}(t,\varepsilon))\psi_{31}^1+c_{21}(t,\varepsilon)\psi_{21}^1+$$

$$+q_{31}^1(t,\varepsilon,\theta)-u_{31}^1(t,\varepsilon)-\varepsilon v_{31}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{32}^1}{dt}=i(m_{23}(t,\varepsilon)-m_{13}(t,\varepsilon))\psi_{32}^1+c_{21}(t,\varepsilon)(\psi_{21}^1-\psi_{33}^1)+c_{32}(t,\varepsilon)\psi_{31}^1+$$

$$+q_{32}^1(t,\varepsilon,\theta)-u_{32}^1(t,\varepsilon)-\varepsilon v_{32}^1(t,\varepsilon,\theta),$$

$$\frac{d\psi_{33}^1}{dt}=c_{21}(t,\varepsilon)\psi_{23}^1+q_{33}^1(t,\varepsilon,\theta)-u_{33}^1(t,\varepsilon)-\varepsilon v_{33}^1(t,\varepsilon,\theta).$$

Define $\psi_{jk}^1$, $u_{jk}^1$, $v_{jk}^1$ by the following expression:

$$\psi_{21}^1(t,\varepsilon,\theta)=-\sum_{n=-\infty}^{\infty}\frac{\Gamma_n[q_{21}^1(t,\varepsilon,\theta)]}{i(m_{13}(t,\varepsilon)-m_{12}(t,\varepsilon)-n\varphi(t,\varepsilon))}\,e^{in\theta(t,\varepsilon)},$$

$$u_{21}^1(t,\varepsilon)\equiv 0,$$

$$v_{21}^1(t,\varepsilon,\theta)=\frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{21}^1(t,\varepsilon,\theta)]}{i(m_{13}(t,\varepsilon)-m_{12}(t,\varepsilon)-n\varphi(t,\varepsilon))}\right)e^{in\theta(t,\varepsilon)},$$

$$\psi_{11}^1(t,\varepsilon,\theta)=\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{\Gamma_n[q_{11}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\,e^{in\theta(t,\varepsilon)},$$

$$u_{11}^1(t,\varepsilon)=\Gamma_0[q_{11}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)],$$

$$v_{11}^1(t,\varepsilon,\theta)=-\frac{1}{\varepsilon}\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}\frac{d}{dt}\left(\frac{\Gamma_n[q_{11}^1(t,\varepsilon,\theta)-c_{32}(t,\varepsilon)\psi_{21}^1(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\right)e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{31}(t,\varepsilon,\theta) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[q^1_{31}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi^1_{21}(t,\varepsilon,\theta)]}{i(m_{23}(t,\varepsilon) - m_{12}(t,\varepsilon) - n\varphi(t,\varepsilon))}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{31}(t,\varepsilon) \equiv 0,$$

$$v^1_{31}(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{31}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi^1_{21}(t,\varepsilon,\theta)]}{i(m_{13}(t,\varepsilon) - m_{12}(t,\varepsilon) - n\varphi(t,\varepsilon))}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{23}(t,\varepsilon,\theta) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[q^1_{23}(t,\varepsilon,\theta)]}{i(m_{13}(t,\varepsilon) - m_{23}(t,\varepsilon) - n\varphi(t,\varepsilon))}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{23}(t,\varepsilon) \equiv 0,$$

$$v^1_{23}(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{23}(t,\varepsilon,\theta)]}{i(m_{13}(t,\varepsilon) - m_{23}(t,\varepsilon) - n\varphi(t,\varepsilon))}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{33}(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n[q^1_{33}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{33}(t,\varepsilon) = \Gamma_0[q^1_{33}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)],$$

$$v^1_{33}(t,\varepsilon,\theta) = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{33}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{22}(t,\varepsilon,\theta) = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n[q^1_{22}(t,\varepsilon,\theta) + c_{32}(t,\varepsilon)\psi^1_{21}(t,\varepsilon,\theta) - c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{22}(t,\varepsilon) = \Gamma_0[q^1_{22}(t,\varepsilon,\theta) + c_{32}(t,\varepsilon)\psi^1_{21}(t,\varepsilon,\theta) - c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)],$$

$$v^1_{22}(t,\varepsilon,\theta) = -\frac{1}{\varepsilon}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{22}(t,\varepsilon,\theta) + c_{32}(t,\varepsilon)\psi^1_{21}(t,\varepsilon,\theta) - c_{21}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{in\varphi(t,\varepsilon)}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{32}(t,\varepsilon,\theta) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[q^1_{32}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)(\psi^1_{22} - \psi^1_{33}) + c_{32}(t,\varepsilon)\psi^1_{31}]}{i(m_{23}(t,\varepsilon) - m_{13}(t,\varepsilon) - n\varphi(t,\varepsilon))}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{32}(t,\varepsilon) \equiv 0,$$

$$v^1_{32}(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{32}(t,\varepsilon,\theta) + c_{21}(t,\varepsilon)(\psi^1_{22} - \psi^1_{33}) + c_{32}(t,\varepsilon)\psi^1_{31}]}{i(m_{23}(t,\varepsilon) - m_{13}(t,\varepsilon) - n\varphi(t,\varepsilon))}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{13}(t,\varepsilon,\theta) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[q^1_{13}(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{i(m_{12}(t,\varepsilon) - m_{23}(t,\varepsilon) - n\varphi(t,\varepsilon))}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{13}(t,\varepsilon) \equiv 0,$$

$$v^1_{13}(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{13}(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)\psi^1_{23}(t,\varepsilon,\theta)]}{i(m_{12}(t,\varepsilon) - m_{23}(t,\varepsilon) - n\varphi(t,\varepsilon))}\right) e^{in\theta(t,\varepsilon)},$$

$$\psi^1_{12}(t,\varepsilon,\theta) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[q^1_{12}(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)(\psi^1_{22} - \psi^1_{11}) - c_{21}(t,\varepsilon)\psi^1_{13}]}{i(m_{12}(t,\varepsilon) - m_{13}(t,\varepsilon) - n\varphi(t,\varepsilon))}\, e^{in\theta(t,\varepsilon)},$$

$$u^1_{12}(t,\varepsilon) \equiv 0,$$

$$v^1_{12}(t,\varepsilon,\theta) = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty} \frac{d}{dt}\left(\frac{\Gamma_n[q^1_{12}(t,\varepsilon,\theta) - c_{32}(t,\varepsilon)(\psi^1_{22} - \psi^1_{11}) - c_{21}(t,\varepsilon)\psi^1_{13}]}{i(m_{12}(t,\varepsilon) - m_{13}(t,\varepsilon) - n\varphi(t,\varepsilon))}\right) e^{in\theta(t,\varepsilon)}.$$

All the elements of matrix $U_1$ belongs to the class $S(m;\varepsilon_0)$. All the elements of matrix $\Psi_1$ belongs to the class $F(m;\varepsilon_0;\theta)$. All the elements of matrix $V_1$ belongs to the class $F(m-1;\varepsilon_0;\theta)$.

All the equations (12) are considered similarly to equations (11), and so the matrices $\Psi_l$, $U_l$, $V_l$ $(l=\overline{1,q})$ are determined. And also all the elements of matrix $\Psi_l$ belongs to the class $F(m;\varepsilon_0;\theta)$, all the elements of matrix $U_l$ belongs to the class $S(m;\varepsilon_0)$, all the elements of matrix $V_l$ belongs to the class $F(m-1;\varepsilon_0;\theta)$ $(l=\overline{1,q})$. Matrix $W$ are determined from the equations (13).

Lemma 1 are proved.

**Problem solving method and basic results.**

We seek the transformation of the kind:

$$x=(E_3+\mu\Psi(t,\varepsilon,\theta,\mu))y, \tag{15}$$

$y=\mathrm{colon}(y_1,y_2,y_3)$, $E_3$ – unit matrix of third order,

$$\Psi(t,\varepsilon,\theta,\mu)=\begin{pmatrix} 0 & \psi_{12}(t,\varepsilon,\theta,\mu) & \psi_{13}(t,\varepsilon,\theta,\mu) \\ 0 & 0 & \psi_{23}(t,\varepsilon,\theta,\mu) \\ 0 & 0 & 0 \end{pmatrix},$$

$\psi_{jk}\in F(m_1;\varepsilon_1;\theta)$ $(0\le l_1\le m;\ 0\le\varepsilon_1<\varepsilon_0)$, which leads the system (6) to the kind:

$$\frac{dy}{dt}=(B(t,\varepsilon)+\mu D(t,\varepsilon,\theta,\mu))\,y, \tag{16}$$

where

$$D(t,\varepsilon,\theta,\mu)=\begin{pmatrix} d_{11}(t,\varepsilon,\theta,\mu) & 0 & 0 \\ d_{21}(t,\varepsilon,\theta,\mu) & d_{22}(t,\varepsilon,\theta,\mu) & 0 \\ d_{31}(t,\varepsilon,\theta,\mu) & d_{32}(t,\varepsilon,\theta,\mu) & d_{33}(t,\varepsilon,\theta,\mu) \end{pmatrix}.$$

We substitute the expression (15) into system (6), and require that the transformed system has the kind (16). We obtain the next system of the differential equations for detemining $\psi_{12},\psi_{13},\psi_{23}$:

$$\begin{aligned}\tfrac{d\psi_{12}}{dt}&=K_{12}(t,\varepsilon,\theta,\psi_{12},\psi_{13},\psi_{23},\mu),\\ \tfrac{d\psi_{13}}{dt}&=K_{13}(t,\varepsilon,\theta,\psi_{12},\psi_{13},\psi_{23},\mu),\\ \tfrac{d\psi_{23}}{dt}&=K_{23}(t,\varepsilon,\theta,\psi_{12},\psi_{13},\psi_{23},\mu),\end{aligned} \tag{17}$$

where

$$\begin{gathered}K_{12}=(b_{11}(t,\varepsilon)-b_{22}(t,\varepsilon))\psi_{12}-b_{32}(t,\varepsilon)\psi_{13}+p_{12}(t,\varepsilon,\theta)+\\ ++\mu b_{21}(t,\varepsilon)\psi_{12}^2+\mu b_{32}(t,\varepsilon)\psi_{12}\psi_{23}-\\ -\mu^2p_{21}(t,\varepsilon,\theta)\psi_{12}^2+\mu^2b_{31}(t,\varepsilon)\psi_{12}^2\psi_{23}+\mu^2p_{32}(t,\varepsilon,\theta)\psi_{12}\psi_{23}+\\ +\mu^2b_{31}(t,\varepsilon)\psi_{12}\psi_{13}+\mu^2p_{32}(t,\varepsilon,\theta)\psi_{13}+\mu^3p_{31}(t,\varepsilon,\theta)\psi_{12}^2\psi_{23}+\mu^3p_{31}(t,\varepsilon,\theta)\psi_{12}\psi_{13},\\ K_{13}=(b_{11}(t,\varepsilon)-b_{33}(t,\varepsilon))\psi_{13}+p_{13}(t,\varepsilon,\theta)+\mu(p_{11}(t,\varepsilon,\theta)-p_{33}(t,\varepsilon,\theta))\psi_{13}+\\ +\mu p_{12}(t,\varepsilon,\theta)\psi_{23}-\mu b_{13}(t,\varepsilon)\psi_{13}^2-\mu b_{32}(t,\varepsilon)\psi_{13}\psi_{23}-\\ -\mu^2p_{31}(t,\varepsilon,\theta)\psi_{13}^2-\mu^2p_{32}(t,\varepsilon,\theta)\psi_{13}\psi_{23},\\ K_{23}=(b_{22}(t,\varepsilon)-b_{33}(t,\varepsilon))\psi_{23}+b_{21}(t,\varepsilon)\psi_{13}+p_{23}(t,\varepsilon,\theta)+\mu p_{21}(t,\varepsilon,\theta)\psi_{12}+\\ +\mu(p_{22}(t,\varepsilon,\theta)-p_{33}(t,\varepsilon,\theta))\psi_{23}-\mu b_{31}(t,\varepsilon)\psi_{13}\psi_{23}-\\ -\mu b_{32}(t,\varepsilon)\psi_{23}^2-\mu^2p_{31}(t,\varepsilon,\theta)\psi_{13}\psi_{23}-\mu^2p_{32}(t,\varepsilon,\theta)\psi_{23}^2.\end{gathered}$$

In this case $d_{jk}(t,\varepsilon,\theta,\mu)$ $(j\ge k)$ has a kind:

$$\begin{gathered}d_{31}(t,\varepsilon,\theta)=p_{31}(t,\varepsilon,\theta),\\ d_{32}(t,\varepsilon,\theta,\mu)=p_{32}(t,\varepsilon,\theta)+b_{31}(t,\varepsilon)\psi_{12}+\mu p_{31}(t,\varepsilon,\theta)\psi_{12},\\ d_{33}(t,\varepsilon,\theta,\mu)=p_{33}(t,\varepsilon,\theta)+b_{31}(t,\varepsilon)\psi_{13}+b_{32}(t,\varepsilon)\psi_{23}+\mu(p_{31}(t,\varepsilon,\theta)\psi_{13}+p_{32}(t,\varepsilon,\theta)\psi_{23}),\\ d_{21}(t,\varepsilon,\theta,\mu)=p_{21}(t,\varepsilon,\theta)-b_{31}(t,\varepsilon)\psi_{13}-\mu p_{31}(t,\varepsilon,\theta)\psi_{23},\\ d_{22}(t,\varepsilon,\theta,\mu)=p_{22}(t,\varepsilon,\theta)-b_{21}(t,\varepsilon)\psi_{12}-b_{32}(t,\varepsilon)\psi_{23}+\mu p_{21}(t,\varepsilon,\theta)\psi_{12}-\mu d_{32}(t,\varepsilon,\theta,\mu)\psi_{23},\\ d_{11}(t,\varepsilon,\theta,\mu)=p_{11}(t,\varepsilon,\theta)-b_{21}(t,\varepsilon)\psi_{12}-b_{31}(t,\varepsilon)\psi_{13}-\mu(d_{21}(t,\varepsilon,\theta,\mu)\psi_{12}+p_{31}(t,\varepsilon,\theta)\psi_{13}).\end{gathered} \tag{18}$$

**The case 1**. $|\mathrm{Re}(b_{jj}(t,\varepsilon)-b_{kk}(t,\varepsilon))|\ge\gamma>0$ $(j\ne k)$.

From the results of the paper [6] follows the theorems.

**Theorem 1.** *In the case 1 there exists* $\mu_1 \in (0, \mu_0)$ *such that for all* $\mu \in (0, \mu_1)$ *there exists unique particular solution* $\psi_{jk}(t, \varepsilon, \theta, \mu)$ $(j < k)$ *of the system (17), all the components of which belongs to the class* $F(m; \varepsilon_0; \theta)$.

**Theorem 2.** *In the case 1 there exists* $\mu_1 \in (0, \mu_0)$ *such that for all* $\mu \in (0, \mu_1)$ *there exists the transformation of the kind (15), whose coefficients* $\psi_{jk}(t, \varepsilon, \theta, \mu)$ $(j < k)$ *belongs to the class* $F(m; \varepsilon_0; \theta)$, *which leads the system (6) to the triangular kind (16), where* $d_{jk}(t, \varepsilon, \theta, \mu)$ $(j \geq k)$ *are determined by the formulas (18).*

**The case 2**. $b_{jj}(t, \varepsilon) - b_{kk}(t, \varepsilon) = im_{jk}(t, \varepsilon)$, $m_{jk} \in \mathbb{R}$,
$\inf\limits_{G(\varepsilon_0)} |m_{jk}(t, \varepsilon) - n\varphi(t, \varepsilon)| \geq \gamma > 0 \quad \forall n \in \mathbb{Z}$.

Together with the system (17) we consider the auxiliary system:

$$\begin{aligned} \varphi(t, \varepsilon)\tfrac{d\psi_{12}}{d\theta} &= K_{12}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu), \\ \varphi(t, \varepsilon)\tfrac{d\psi_{13}}{d\theta} &= K_{13}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu), \\ \varphi(t, \varepsilon)\tfrac{d\psi_{23}}{d\theta} &= K_{23}(t, \varepsilon, \theta, \psi_{12}, \psi_{13}, \psi_{23}, \mu), \end{aligned} \tag{19}$$

where $\varphi(t, \varepsilon)$ – function in the definition of the class $F(m; \varepsilon_0; \theta)$, and $t, \varepsilon$ are considered as constant. Using the method of the small parameter of Poincarais [7], we construct the partial sums of the series in degrees of the small parameter representing the $2\pi$-periodic with respect to $\theta$ solution of the system (19):

$$\psi^*_{jk}(t, \varepsilon, \theta, \mu) = \psi^0_{jk}(t, \varepsilon, \theta) + \mu\psi^1_{jk}(t, \varepsilon, \theta) + \ldots + \mu^{2q-1}\psi^{2q-1}_{jk}(t, \varepsilon, \theta), \tag{20}$$

where $\psi^s_{jk}(t, \varepsilon, \theta)$ $(s = \overline{0, 2q-1})$ – $2\pi$-periodic with respect to $\theta$ functions. Regarding these functions, we obtain the chain of the system of the differential equations:

$$\begin{aligned} \varphi(t, \varepsilon)\tfrac{d\psi^0_{12}}{d\theta} &= im_{12}(t, \varepsilon)\psi^0_{12} - b_{32}(t, \varepsilon)\psi^0_{13} + p_{12}(t, \varepsilon, \theta), \\ \varphi(t, \varepsilon)\tfrac{d\psi^0_{13}}{d\theta} &= im_{13}(t, \varepsilon)\psi^0_{13} + p_{13}(t, \varepsilon, \theta), \\ \varphi(t, \varepsilon)\tfrac{d\psi^0_{23}}{d\theta} &= im_{23}(t, \varepsilon)\psi^0_{23} + b_{21}(t, \varepsilon)\psi^0_{13} + p_{23}(t, \varepsilon, \theta), \end{aligned} \tag{21}$$

$$\begin{gathered} \varphi(t, \varepsilon)\tfrac{d\psi^s_{12}}{d\theta} = im_{12}(t, \varepsilon)\psi^s_{12} - b_{32}(t, \varepsilon)\psi^s_{13} + \\ + P^s_{12}(t, \varepsilon, \theta, \psi^0_{12}, \psi^0_{13}, \psi^0_{23}, \ldots, \psi^{s-1}_{12}, \psi^{s-1}_{13}, \psi^{s-1}_{23}), \\ \varphi(t, \varepsilon)\tfrac{d\psi^s_{13}}{d\theta} = im_{13}(t, \varepsilon)\psi^0_{13} + \\ + Q^s_{13}(t, \varepsilon, \theta, \psi^0_{12}, \psi^0_{13}, \psi^0_{23}, \ldots, \psi^{s-1}_{12}, \psi^{s-1}_{13}, \psi^{s-1}_{23}), \\ \varphi(t, \varepsilon)\tfrac{d\psi^s_{23}}{d\theta} = im_{23}(t, \varepsilon)\psi^s_{23} + b_{21}(t, \varepsilon)\psi^s_{13} + \\ + R^s_{23}(t, \varepsilon, \theta, \psi^0_{12}, \psi^0_{13}, \psi^0_{23}, \ldots, \psi^{s-1}_{12}, \psi^{s-1}_{13}, \psi^{s-1}_{23}), \ s = 1, 2, \ldots, 2q-1. \end{gathered} \tag{22}$$

$P^s_{12}, Q^s_{13}, R^s_{23}$ – polynomials from $\psi^0_{12}, \ldots, \psi^{s-1}_{23}$ with coefficients from the class $F(m; \varepsilon_0; \theta)$.

Consider a generating system (21). In the case 2 this system has unique $2\pi$-periodic with respect to $\theta$ solution:

$$\psi^0_{13}(t, \varepsilon, \theta) = \sum_{n=-\infty}^{\infty} \psi^0_{13,n}(t, \varepsilon)\exp(in\theta),$$

where

$$\psi^0_{13,n}(t, \varepsilon) = -\frac{\Gamma_n[p_{13}(t, \varepsilon, \theta)]}{i(m_{13}(t, \varepsilon) - n\varphi(t, \varepsilon))},$$

$$\psi_{12}^0(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}\psi_{12,n}^0(t,\varepsilon)\exp(in\theta),$$

where

$$\psi_{12,n}^0(t,\varepsilon)=-\frac{\Gamma_n[p_{12}(t,\varepsilon,\theta)]-b_{32}(t,\varepsilon)\psi_{13,n}^0(t,\varepsilon)}{i(m_{12}(t,\varepsilon)-n\varphi(t,\varepsilon))},$$

$$\psi_{23}^0(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}\psi_{23,n}^0(t,\varepsilon)\exp(in\theta),$$

where

$$\psi_{23,n}^0(t,\varepsilon)=-\frac{\Gamma_n[p_{23}(t,\varepsilon,\theta)]+b_{21}(t,\varepsilon)\psi_{13,n}^0(t,\varepsilon)}{i(m_{23}(t,\varepsilon)-n\varphi(t,\varepsilon))},$$

and $\psi_{13}^0(t,\varepsilon,\theta)$, $\psi_{12}^0(t,\varepsilon,\theta)$, $\psi_{23}^0(t,\varepsilon,\theta)$ belongs to the class $F(m;\varepsilon_0;\theta)$.

Similarly, all systems in the chain (22) also have a unique $2\pi$-periodic with respect to $\theta$ solutions, and all components of theese solutions belongs to the class $F(m;\varepsilon_0;\theta)$.

Consequently, the functions $\psi_{jk}^*(t,\varepsilon,\theta,\mu)$ belongs to the class $F(m;\varepsilon_0;\theta)$ also.

We make in the system (17) the substitution:

$$\psi_{jk}=\psi_{jk}^*(t,\varepsilon,\theta,\mu)+\xi_{jk}\ (j<k). \tag{23}$$

We obtain:

$$\frac{d\xi_{12}}{dt}=im_{12}(t,\varepsilon)\xi_{12}-b_{32}(t,\varepsilon)\xi_{13}+\varepsilon g_{12}(t,\varepsilon,\theta,\mu)+$$
$$+\mu^{2q}c_{12}(t,\varepsilon,\theta,\mu)+\left(\sum_{l=1}^{q}b_{12l}(t,\varepsilon,\theta)\mu^l\right)\xi_{12}+\left(\sum_{l=1}^{q}c_{12l}(t,\varepsilon,\theta)\mu^l\right)\xi_{13}+$$
$$+\left(\sum_{l=1}^{q}d_{12l}(t,\varepsilon,\theta)\mu^l\right)\xi_{23}+\mu^{q+1}\left(\alpha_{12}(t,\varepsilon,\theta,\mu)\xi_{12}+\beta_{12}(t,\varepsilon,\theta,\mu)\xi_{13}+\right.$$
$$\left.+\gamma_{12}(t,\varepsilon,\theta,\mu)\xi_{23}\right)+\mu\Xi_{12}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),$$
$$\frac{d\xi_{13}}{dt}=im_{13}(t,\varepsilon)\xi_{13}+\varepsilon g_{13}(t,\varepsilon,\theta,\mu)+$$
$$+\mu^{2q}c_{13}(t,\varepsilon,\theta,\mu)+\left(\sum_{l=1}^{q}b_{13l}(t,\varepsilon,\theta)\mu^l\right)\xi_{12}+\left(\sum_{l=1}^{q}c_{13l}(t,\varepsilon,\theta)\mu^l\right)\xi_{13}+$$
$$+\left(\sum_{l=1}^{q}d_{13l}(t,\varepsilon,\theta)\mu^l\right)\xi_{23}+\mu^{q+1}\left(\alpha_{13}(t,\varepsilon,\theta,\mu)\xi_{12}+\beta_{13}(t,\varepsilon,\theta,\mu)\xi_{13}+\right.$$
$$\left.+\gamma_{13}(t,\varepsilon,\theta,\mu)\xi_{23}\right)+\mu\Xi_{13}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),$$
$$\frac{d\xi_{23}}{dt}=im_{23}(t,\varepsilon)\xi_{23}+b_{21}(t,\varepsilon)\xi_{13}+\varepsilon g_{23}(t,\varepsilon,\theta,\mu)+$$
$$+\mu^{2q}c_{23}(t,\varepsilon,\theta,\mu)+\left(\sum_{l=1}^{q}b_{23l}(t,\varepsilon,\theta)\mu^l\right)\xi_{12}+\left(\sum_{l=1}^{q}c_{23l}(t,\varepsilon,\theta)\mu^l\right)\xi_{13}+$$
$$+\left(\sum_{l=1}^{q}d_{23l}(t,\varepsilon,\theta)\mu^l\right)\xi_{23}+\mu^{q+1}\left(\alpha_{23}(t,\varepsilon,\theta,\mu)\xi_{12}+\beta_{23}(t,\varepsilon,\theta,\mu)\xi_{13}+\right.$$
$$\left.+\gamma_{23}(t,\varepsilon,\theta,\mu)\xi_{23}\right)+\mu\Xi_{23}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu), \tag{24}$$

where $g_{12},g_{13},g_{23}\in F(m-1;\varepsilon_0;\theta)$, $c_{12},c_{13},c_{23},b_{jkl},c_{jkl},d_{jkl},\alpha_{jk},\beta_{jk},$ $\gamma_{jk}\in F(m;\varepsilon_0;\theta)$ $(j<k)$, $\Xi_{12},\Xi_{13},\Xi_{23}$ – polynomials with respect to $\xi_{12},\xi_{13},\xi_{23}$ with coefficients from the class $F(m;\varepsilon_0;\theta)$, containing terms not lower than second order with respect $\xi_{12},\xi_{13},\xi_{23}$.

We introduce $\xi=\mathrm{colon}(\xi_{12},\xi_{13},\xi_{23})$,

$$A_1(t,\varepsilon)=\begin{pmatrix} im_{12}(t,\varepsilon) & -b_{32}t,\varepsilon) & 0\\ 0 & im_{13}(t,\varepsilon) & 0\\ 0 & b_{21}(t,\varepsilon) & im_{23}(t,\varepsilon)\end{pmatrix},$$

$g(t,\varepsilon,\theta,\mu)=\mathrm{colon}(g_{12}(t,\varepsilon,\theta,\mu),g_{13}(t,\varepsilon,\theta,\mu),g_{23}(t,\varepsilon,\theta,\mu))$ ,
$c(t,\varepsilon,\theta,\mu)=\mathrm{colon}(c_{12}(t,\varepsilon,\theta,\mu),c_{13}(t,\varepsilon,\theta,\mu),c_{23}(t,\varepsilon,\theta,\mu))$ ,

$$K_l(t,\varepsilon,\theta)=\begin{pmatrix} b_{12l}(t,\varepsilon,\theta) & c_{12l}t,\varepsilon,\theta) & d_{12l}t,\varepsilon,\theta) \\ b_{13l}(t,\varepsilon,\theta) & c_{13l}(t,\varepsilon,\theta) & d_{13l}(t,\varepsilon,\theta) \\ b_{23l}(t,\varepsilon,\theta) & c_{23l}(t,\varepsilon,\theta) & d_{23l}(t,\varepsilon,\theta) \end{pmatrix},$$

$$L(t,\varepsilon,\theta,\mu)=\begin{pmatrix} \alpha_{12}(t,\varepsilon,\theta,\mu) & \beta_{12}t,\varepsilon,\theta,\mu) & \gamma_{12}t,\varepsilon,\theta,\mu) \\ \alpha_{13}(t,\varepsilon,\theta,\mu) & \beta_{13}t,\varepsilon,\theta,\mu) & \gamma_{13}t,\varepsilon,\theta,\mu) \\ \alpha_{23}(t,\varepsilon,\theta,\mu) & \beta_{23}t,\varepsilon,\theta,\mu) & \gamma_{23}t,\varepsilon,\theta,\mu) \end{pmatrix},$$

$\Xi(t,\varepsilon,\theta,\xi,\mu)=\mathrm{colon}(\Xi_{12}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),\Xi_{13}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu),$
$\Xi_{23}(t,\varepsilon,\theta,\xi_{12},\xi_{13},\xi_{23},\mu))$ .

Then the system (24) can be written as:

$$\frac{d\xi}{dt}=\left(A_1(t,\varepsilon)+\sum_{l=1}^{q}K_l(t,\varepsilon,\theta)\mu^l\right)\xi+\varepsilon g(t,\varepsilon,\theta,\mu)+\mu^{2q}c(t,\varepsilon,\theta,\mu)+$$
$$+\mu^{q+1}L(t,\varepsilon,\theta,\mu)\xi+\Xi(t,\varepsilon,\theta,\xi,\mu). \tag{25}$$

Based on Lemma 1, using the conditions (8) and the transformation of kind:

$$\xi=\left(E+\sum_{l=1}^{q}\Psi_l(t,\varepsilon,\theta)\mu^l\right)\eta, \tag{26}$$

where $\eta=\mathrm{colon}(\eta_1,\eta_2,\eta_3)$ , we leads the system (25) to the kind:

$$\frac{d\eta}{dt}=\left(A_1(t,\varepsilon)+\sum_{l=1}^{q}U_l(t,\varepsilon)\mu^l\right)\eta+\varepsilon g^1(t,\varepsilon,\theta,\mu)+\mu^{2q}c^1(t,\varepsilon,\theta,\mu)+$$
$$+\varepsilon\left(\sum_{l=1}^{q}V_l(t,\varepsilon,\theta)\mu^l\right)\eta+\mu^{q+1}L^1(t,\varepsilon,\theta,\mu)+\mu H(t,\varepsilon,\theta,\eta,\mu), \tag{27}$$

where $U_l(t,\varepsilon)=\mathrm{diag}(u_{1l}(t,\varepsilon),u_{2l}(t,\varepsilon),u_{3l}(t,\varepsilon))$ , and $u_{jl}(t,\varepsilon)\in S(m;\varepsilon_0)$ $(j=1,2,3;\ l=\overline{1,q})$ .

**Lemma 2.** *Let the system (27) satisfy the next conditions*:
1) *the eigenvalues* $\lambda_j(t,\varepsilon,\mu)$ $(j=1,2,3)$ *of the matrix*

$$U(t,\varepsilon,\mu)=A_1(t,\varepsilon)+\sum_{l=1}^{q}U_l(t,\varepsilon)\mu^l$$

*such that*

$$\inf_{G(\varepsilon_0)}|\mathrm{Re}\lambda_j(t,\varepsilon,\theta)|\geq\gamma_0\mu^{q_0}\ (\gamma_0\geq 0,\ 0<q_0\leq q);$$

2) *for the matrix* $U(t,\varepsilon,\mu)$ *there exists the matrix* $Y(t,\varepsilon,\mu)$ *such that*
a) $\inf\limits_{G(\varepsilon_0)}|\mathrm{det}Y(t,\varepsilon,\mu)|>0,$
b) $Y^{-1}UY=\Lambda(t,\varepsilon,\mu)$ *– diagonal matrix.*

*Then there exists* $\mu_2\in(0,\mu_0)$ , $\varepsilon_1(\mu)\in(0,\varepsilon_0)$ *such that for all* $\mu\in(0,\mu_2)$ *and for all* $\varepsilon\in(0,\varepsilon_1(\mu))$ *there exists the particular solution of the system (27), all the components of which belongs to the class* $F(m-1;\varepsilon_1(\mu);\theta)$ .

**Proof.** Based on condition 2) of Lemma, we make in the system (27) the substitution:

$$\eta=\frac{\varepsilon+\mu^{2q}}{\mu^{q_0}}\ Y(t,\varepsilon,\mu)\chi. \tag{28}$$

We obtain:

$$\frac{d\chi}{dt}=\Lambda(t,\varepsilon,\mu)\chi+\frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\ g^2(t,\varepsilon,\theta,\mu)+\frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\ c^2(t,\varepsilon,\theta,\mu)+$$
$$+\varepsilon A_2(t,\varepsilon,\theta,\mu)\chi+\mu^{q+1}C(t,\varepsilon,\theta,\mu)\chi+\frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\ X(t,\varepsilon,\theta,\chi,\mu), \tag{29}$$

where elements of vector $g^2$ and matrix $A_2$ belongs to the class $F(m-1;\varepsilon_0;\theta)$, elements of vector $c^2$ and matrix $C$ belongs to the class $F(m;\varepsilon_0;\theta)$, elements of vector-function $X$ belongs to the class $F(m;\varepsilon_0;\theta)$ in respect to $t,\varepsilon,\theta$ and polynomials in respect to elements of vector $\chi$.

Together with the system (29) we consider the linear nonhomogeneous system:

$$\frac{d\chi^0}{dt} = \Lambda(t,\varepsilon,\mu)\chi^0 + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\, g^2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\, c^2(t,\varepsilon,\theta,\mu). \tag{30}$$

From the results of the paper [6], based on conditions 1) of Lemma, we obtain, that there exists particular solution $\chi^0(t,\varepsilon,\theta,\mu)$ of the system (30), all elements of which belongs to the class $F(m-1;\varepsilon_0;\theta)$, and there exists $K\in(0,+\infty)$ such that:

$$\|\chi^0\|^*_{F(m-1;\varepsilon_0;\theta)} \leq \frac{K}{\gamma\mu^{q_0}}\left(\frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\,\|g^2\|^*_{F(m-1;\varepsilon_0;\theta)} + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\,\|c^2\|^*_{F(m-1;\varepsilon_0;\theta)}\right) <$$
$$< \frac{K}{\gamma}\left(\|g^2\|^*_{F(m-1;\varepsilon_0;\theta)} + \|c^2\|^*_{F(m-1;\varepsilon_0;\theta)}\right).$$

We construct the process of succesive approximation, defininng as initial approximation $\chi^0$, and subsequent approximations defining as solutions from the class $F(m-1;\varepsilon_0;\theta)$ of the systems:

$$\frac{d\chi^{j+1}}{dt} = \Lambda(t,\varepsilon,\mu)\chi^{j+1} + \frac{\varepsilon\mu^{q_0}}{\varepsilon+\mu^{2q}}\, g^2(t,\varepsilon,\theta,\mu) + \frac{\mu^{2q+q_0}}{\varepsilon+\mu^{2q}}\, c^2(t,\varepsilon,\theta,\mu)+$$
$$+\varepsilon A_2(t,\varepsilon,\theta,\mu)\chi^j + \mu^{q+1}C(t,\varepsilon,\theta,\mu)\chi^j + \frac{\varepsilon+\mu^{2q}}{\mu^{q_0-1}}\, X(t,\varepsilon,\theta,\chi^j,\mu),\ j=0,1,2,.... \tag{31}$$

Using an usual techniques contraction mapping principle [8] it is easy to show that there exists $\mu_3\in(0,\mu_0)$ and $\varepsilon_1(\mu)=K_2\mu$, where $K_2$ – sufficiently small constant, such that for all $\mu\in(0,\mu_3)$ and for all $\varepsilon\in(0,\varepsilon_1(\mu))$ the process (31) converges to the solution $\chi(t,\varepsilon,\theta,\mu)$ of the system (29), and all components of this solution belongs to the class $F(m-1;\varepsilon_1(\mu);\theta)$.

Lemma 2 are proved.

The following statements are an immediate consequences of Lemma 2.

**Lemma 3.** *Let the system (17) be such that:*

1) *conditions (8) are satisfies;*

2) *for the system (27), obtained from the system (17) using the transformation (23), (26), all conditions of Lemma 2 are satisfies.*

*Then there exists $\mu_4\in(0,\mu_0)$, $\varepsilon_2(\mu)\in(0,\varepsilon_0)$ such that for all $\mu\in(0,\mu_4)$ and for all $\varepsilon\in(0,\varepsilon_2(\mu))$ there exists the particular solution of the system (17), all the components of which belongs to the class $F(m-1;\varepsilon_2(\mu);\theta)$.*

**Theorem 3.** *Let the system (17) satisfies all conditions of Lemma 3. Then in the case 2 there exists $\mu_4\in(0,\mu_0)$, $\varepsilon_2(\mu)\in(0,\mu_0)$ such that for all $\mu\in(0,\mu_4)$* and for all $\varepsilon\in(0,\varepsilon_2(\mu))$ *there exists the transformation of the kind (15), whose coefficients $\psi_{jk}(t,\varepsilon,\theta,\mu)$ $(j<k)$ belongs to the class $F(m-1;\varepsilon_2(\mu);\theta)$, which leads the system (6) to a triangular kind (16), where $d_{jk}(t,\varepsilon,\theta,\mu)$ $(j\geq k)$ are determines by the formulas (18).*

**Conclusions.** Thus, for the system (2) the conditions of the existence of the transformation with coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, which leads it to triangular kind, are obtained in the non-resonant cases.

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**Резонансты емес жағдайда осцилляциялы типті коэффициентті сызықты дифференциалдық теңдеулер жүйесін үшбұрышты түрге келтіру туралы**

**Аннотация:** Коэффициенттері мен жиіліктері баяу өзгереіп, абсолютті және бірқалыпты жинақталатын Фурье қатарлары түрінде өрнектелетін сызықты біртекті дифференциалдық теңдеулер жүйесі үшін осы жүйені резонансты емес жағдайда үшбұрышты түрге келтіретін түрлендірудің бар болу шарттары алынған.

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**О приведении линейной системы дифференциальных уравнений с коэффициентами осциллирующего типа к треугольному виду в нерезонансном случае**

**Аннотация:** Для линейной однородной дифференциальной системы, коэффициенты которой представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования преобразования, приводящего эту систему к треугольному виду в нерезонансном случае.

**Ключевые слова:** линейные дифференциальные системы, ряды Фурье.

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